		Player B			
		I	II	III	IV
Player A	1	18	4	6	4
	2	6	2	13	7
	3	11	5	17	3
	4	7	6	12	2

[AIMS (MBA) 2002]

32. (a) Use the Dominance principle and solve the game :

		B		
		1	2	3
A	I	1	−3	−2
	II	0	−4	2
	III	−5	2	3

[VTU 2003]

(b) Solve the following game using graphical method.

		B		
		1	2	3
A	I	3	−1	0
	II			
	III	2	1	−1

[VTU 2003]

MODEL OBJECTIVE QUESTIONS

1. Two-person zero-sum game means that the
(a) sum of losses to one player equals the sum of gains to other.
(b) sum of losses to one player is not equal to the sum of gains to other.
(c) both (a) and (b). (d) none of the above.

2. Game theory models are classified by the
(a) number of players. (b) sum of all payoffs. (c) number of strategies. (d) all of the above.

3. A game is said to be fair, if
(a) both upper and lower values of the game are same and zero.
(b) upper and lower values of the game are not equal.
(c) upper value is more than lower value of the game.
(d) none of the above.

4. What happens when maximin and minimax values of the game are same?
(a) No solution exists. (b) Solution is mixed. (c) Saddle point exists. (d) None of the above.

5. A mixed strategy game can be solved by
(a) algebraic method. (b) matrix method. (c) graphical method. (d) all of the above.

6. The size of the payoff matrix of a game can be reduced by using the principle of
(a) game inversion. (b) rotation reduction. (c) dominance. (d) game transpose.

7. The payoff value for which each player in a game always selects the same strategy is called the
(a) saddle point. (b) equilibrium point. (c) both (a) and (b). (d) none of the above.

8. Games which involve more than two players are called
(a) conflicting games. (b) negotiable games. (c) n-person games. (d) all of the above.

9. When the sum of gains of one player is equal to the sum of losses to another player in a game, this situation is known as
(a) biased game. (b) zero-sum game. (c) fair game. (d) all of the above.

10. When no saddle point is found in a payoff matrix of a game, the value of the game is then found by
(a) knowing joint probabilities of each row and column combination to calculate expected payoff for that combination and adding all such values.
(b) reducing size of the game to apply algebraic method.
(c) both (a) and (b). (d) none of the above.

Answers

1. (c) **2.** (d) **3.** (a) **4.** (c) **5.** (d) **6.** (c) **7.** (c) **8.** (c) **9.** (b) **10.** (c)

✱✱✱

10

Queueing Systems

(Waiting Line Models)

10.1. INTRODUCTION

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the ***waiting line,*** in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, (*i.e.* remains idle), until additional people arrive. Here the arriving people are called the ***customers*** and the person issuing the tickets is called a ***server.***

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repairman. These examples show that the term *customer* may be interpreted in various number of ways. It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aeroplanes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations (*e.g.* at check out counters of supermarket).

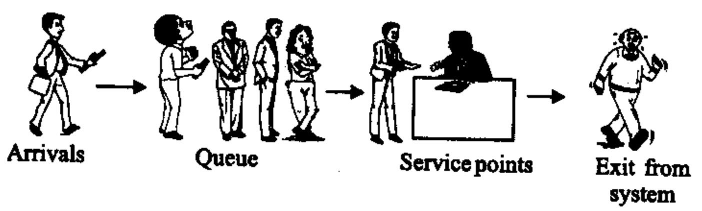

Fig. 10.1 (a). Queueing system with single queue and single service station.

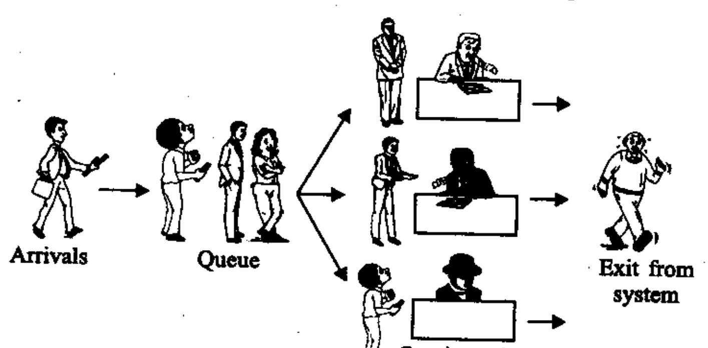

Fig. 10.1 (b). Queueing system with single queue and several service stations.

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single queue as shown in *Fig.* 10.1 or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

Fig. 10.2 illustrates how a machine shop may be thought of as a system of queues forming in front of a number of service centres, the arrows between the centres indicating possible routes for jobs processed in the shop. Arrivals at a service centre are either new jobs coming into the system or jobs, partially processed, from some other service centre. Departures from a service centre may become the arrivals at another service centre or may leave the system entirely, when processing on these items is complete.

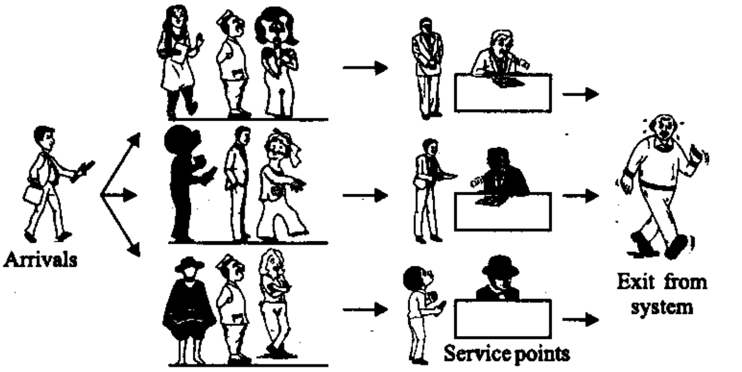

Fig. 10.1 (c). Queueing system with several queues and several service

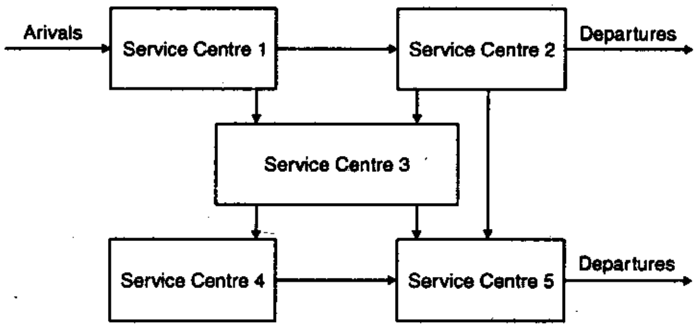

Fig. 10.2. A machine shop as a complex queue.

Queueing theory is concerned with the statistical description of the behaviour of queues with finding, *e.g.*, the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems involving queues, investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system, what effects of various kinds of changes in the system's characteristics would be, and whether, in the light of the costs incurred in the systems, changes should be made to it. A model of the queueing system under study must be constructed in this kind of analysis and the results of queueing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queueing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

10.2. QUEUEING SYSTEM

A queueing system can be completely described by

(*a*) the input (or arrival pattern), (*b*) the service mechanism (or service pattern),

(*c*) the 'queue discipline' and (*d*) customer's behaviour.

(a) The input (or arrival pattern). The input describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction. Thus, the arrival pattern can best be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) or the distribution of number of customers arriving in unit time must be defined.

The present chapter is only dealt with those queueing systems in which the customers arrive in *'Poisson'* or *'completely random'* fashion (see *sec. 10.7-1*). Other types of arrival pattern may also be observed in practice that have been studied in queueing theory. Two such patterns are observed, where

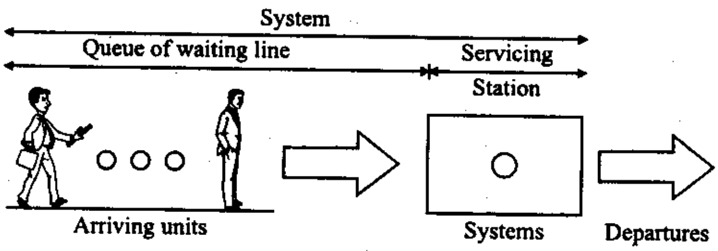

Fig. 10.3. A queueing system with single service station.

(*i*) arrivals are of regular intervals;

(*ii*) there is general distribution (perhaps normal) of time between successive arrivals.

(b) The service mechanism (or service pattern). It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers (*e.g.* machines waiting repair) each with a different service time distribution. Service time may be constant or a random variable. Distributions of service time which are important in practice are ***'negative exponential distribution'*** and the related ***'Erlang (Gamma) distribution'***. Queues with the negative exponential service time distribution are studied in the following sections.

In the present chapter, only those queueing systems are discussed in which the service time follows the *'Exponential and Erlang (Gamma)'* probability distributions (see ***sec. 10.7-1* to *10.7-8***).

(c) The queue discipline. The queue discipline is the rule determining the formation of the queue, the manner of the customer's behaviour while waiting, and the manner in which they are chosen for service. The simplest discipline is ***"first come, first served"***, according to which the customers are served in the order of their arrival. For example, such type of queue discipline is observed *at a ration shop, at cinema ticket windows, at railway stations,* etc. If the order is reversed, we have the ***"last come, first served"*** discipline, as in the case of a big godown the items which come last are taken out first. An extremely difficult queue discipline to handle might be ***"service in random order"*** or ***"might is right"***.

Properties of a queueing system which are concerned with waiting times, in general, depend on queue discipline. For example, the variance of waiting time will be much greater with the queue discipline ***'first come, last served'*** than with ***'first come, first served'***, although mean waiting time will remain unaffected.

The following notations are used for describing the nature of service discipline.

FIFO → First In, First Out or ***FCFS*** → First Come, First Served

LIFO → Last In, First Out or ***FILO*** → First In, Last Out.

SIRO → Service in Random Order

This chapter shall be concerned only with the customers which are served in the order in which they arrive at the service facility, that is, ***'first come, first served'*** discipline.

(d) Customer's behaviour. The customers generally behave in four ways :

(*i*) **Balking.** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.

(*ii*) **Reneging.** This occurs when a waiting customer leaves the queue due to impatience.

(*iii*) **Priorities.** In certain applications some customers are served before others regardless of their order of arrival. These customers have *priority* over others.

(*iv*) **Jockeying.** Customers may *jockey* from one waiting line to another. It may be seen that this occurs in the supermarket.

(e) Size of a Population : The collection of potential customers may be very large or of a moderate size. In a railway booking counter the total number of potential passengers is so large that although theoretically finite it can be regarded as infinity for all practical purposes. The assumption of infinite population is very convenient for analysing a queuing model. Howerver, this assumption is not valid where the customer group is represented by few machines in workshop that require operator facility from time to time. If the population size is finite then the analysis of queueing model becomes more involved.

(f) **Maximum Length of a Queue**: Sometimes only a finite number of customers are allowed to stay in the system although the total number of customers in the population may or may not be finite. For example, a doctor may have appointments with k patients in a day. If the number of patients asking for appointment exceeds k, they are not allowed to join the queue. Thus, although the size of the population is infinite, the maximum number permissible in the system is k.

Q. 1. Explain briefly the main characteristics of queueing system.
2. Describe the fundamental components of a queueing process and give suitable examples. **[C.A. (Nov) 92]**
3. List the factors that constitute the basic elements of a queueing model. For each of these enumerate the alternatives posible. Represent this diagramatically to cover all possible implimentations of a queueing model. **[IGNOU 99 (Dec.)]**
4. What is queueing theory? **[JNTU (MCA III) 2004]**

10.3. QUEUEING PROBLEM

In a specified queueing system, the problem is to determine the following:

(a) **Probability distribution of queue length.** When the nature of probability distributions of the arrival and service patterns is given, the probability distribution of queue length can be obtained. Further, we can also estimate the probability that there is no queue.

(b) **Probability distribution of waiting time of customers.** We can find the time spent by a customer in the queue before the commencement of his service which is called his ***waiting time.*** The total time spent by him in the system is the waiting time plus service time.

(c) **The busy period distribution.** We can estimate the probability distribution of busy periods. If we suppose that the server is free initially and customer arrives, he will be served immediately. During his service time, some more customers will arrive and will be served in their turn. This process will continue in this way until no customer is left unserved and the server becomes free again. Whenever this happens, we say that a **busy period** has just ended. On the other hand, during **idle periods** no customer is present in the system. A busy period and the idle period following it together constitute a *busy cycle.* The study of the busy period is of great interest in cases where technical features of the server and his capacity for continuous operations must be taken into account.

10.4. TRANSIENT AND STEADY STATES

Queueing theory analysis involves the study of a system's behaviour over time. ***A system is said to be in "transient state" when its operating characteristics (behaviour) are dependent on time.*** This usually occurs at the early stages of the operation of the system where its behaviour is still dependent on the initial conditions. However, since we are mostly interested in the *"long run"* behaviour of the system, mainly the attention has been paid toward *"steady state"* results.

A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $[dP_n(t)/dt]$ or $P_n'(t)$. Then the queueing system is said to become *'stable'* eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes ($t \to \infty$). Mathematically, in steady state

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of } t) \Rightarrow \lim_{t \to \infty} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} \Rightarrow \lim_{t \to \infty} P_n'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. In fact, in this case the queue length will increase with time and theoretically it could build up to infinity. Such case is called the ***"explosive state".***

In this chapter, only the steady state analysis will be considered. We shall not treat the *'transient'* and *'explosive'* states.

Q. 1. What is queueing problem ? Explain queueing system, transient and steady state. **[Garhwal M.Sc. (Stat.) 96]**

2. What is a queueing theory problem ? Describe the advantages of queueing theory to a business executive with a view to persuading him to make use of the same in management. **[Garhwal M.Sc. (Stat.) 95]**

3. What do you understand by a queue ? Give some important applications of queueing theory. **[Garhwal M.Sc. (Stat.) 92, 91]**

4. Write an essay on various characteristics of a queueing system. **[Virbhadra 2000]**

5. Discuss the stationery state of the queue system. **[JNTU (Mech. & Prod.) 2004]**

10.5. A LIST OF SYMBOLS

Unless otherwise stated, the following symbols and terminology will be used henceforth in connection with the queueing models. The reader is reminded that a queueing system is defined to include the *queue* and the *service stations* both. (see *Fig.* 10.3).

n = number of units in the system

$P_n(t)$ = transient state probability that exactly n calling units are in the queueing system at time t

E_n = the state in which there are n calling units in the system

P_n = steady state probability of having n units in the system

λ_n = mean arrival rate (expected number of arrivals per unit time) of customers (when n units are present in the system)

μ_n = mean service rate (expected number of customers served per unit time when there are n units in the system)

λ = mean arrival rate when λ_n is constant for all n

μ = mean service rate when μ_n is constant for all $n \geq 1$

s = number of parallel service stations

$\rho = \lambda/\mu s$ = traffic intensity (or utilization factor) for servers facility, that is, the expected fraction of time the servers are busy

$\phi_T(n)$ = probability of n services in time T, given that servicing is going on throughout T

Line length (or queue size)
= number of customers in the queueing system

Queue length
= line length (or queue size)–(number of units being served)

$\psi(w)$ = probability density function (p.d.f.) of waiting time in the system

L_s = expected line length, *i.e.*, expected number of customers in the *system*

L_q = expected queue length, *i.e.*, expected number of customers in the *queue*

W_s = expected waiting time per customer in the system

W_q = expected waiting time per customer in the queue

$(W \mid W > 0)$ = expected waiting time of a customer who has to wait

$(L \mid L > 0)$ = expected length of non-empty queues, *i.e.*, expected number of customers in the queue when there is a queue

$P(W > 0)$ = probability of a customer having to wait for service

$\binom{n}{r}$ = the binomial coefficient ${}^nC_r$.

$$= \frac{n\,!}{r\,!\,(n-r)\,!} = \frac{n\,(n-1)\,\ldots\,(n-r+1)}{r\,!} \quad \text{for } r \text{ and } n \text{ non-negative integers } (r \leq n).$$

10.6. TRAFFIC INTENSITY (OR UTILIZATION FACTOR)

An important measure of a simple queue $(M \mid M \mid 1)$ is its *traffic intensity*, where

$$\text{Traffic intensity } (\rho) = \frac{\text{mean arrival rate}}{\text{mean service rate}} = \frac{\lambda}{\mu}$$

i.e., $$\rho = \frac{1/\mu}{1/\lambda} = \frac{\text{Mean service time}}{\text{Mean inter-arrival time}}$$

The unit of traffic intensity is *Erlang.*

Here it should be noted carefully that a necessary condition for a system to have settled down to steady state is that $\rho < 1$ or $\lambda/\mu < 1$ or $\lambda < \mu$, *i.e., arrival rate < service rate.*

If this is not so, *i.e.,* $\rho > 1$, the arrival rate will be greater than the service rate and consequently, the number of units in the queue tends to increase indefinitely as the time passes on, provided the rate of service is not affected by the length of queue.

10.7. PROBABILITY DISTRIBUTIONS IN QUEUEING SYSTEMS

The arrival pattern of customers at a queueing system varies between one system and another, but one pattern of common occurrence in practice, which turns out to be relatively easy to deal with mathematically, is that of ***'completely random arrivals'***. This phrase means something quite specific, and we discuss what does it mean before dealing in the subsequent sections with a variety of queueing systems. In particular, we show that, if arrivals are *'completely random'*, the number of arrivals in unit time has a *Poisson distribution*, and the intervals between successive arrivals are distributed *negative exponentially.*

10.7-1. Distribution of Arrivals 'The Poisson Process' (Pure Birth Process)

In many situations the objective of an analysis consists of merely observing the number of customers that enter the system. The model in which only arrivals are counted and no departures take place are called *pure birth models.* The term *'birth'* refers to the arrival of a new calling unit in the system, and the *'death'* refers to the departure of a served unit. As such *pure birth* models are not of much importance so far as their applicability to real life situation is concerned, but these are very important in the understanding of completely random arrival problems. **[Bhubneshwar (IT) 2004]**

Theorem 10.1. (***Arrival Distribution Theorem***). *If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time-interval follows a Poisson distribution.*

[Agra 99; Meerut (Stat.) 98; Garhwal M.Sc. (Math.) 94; Raj. Univ. (Math) 93]

Proof. In order to derive the arrival distribution in queues, we make the following three assumptions (sometimes called the ***axioms***).

1. Assume that there are n units in the system at time t, and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda\Delta t + O(\Delta t)$, where λ is the arrival rate independent of t and $O(\Delta t)$ includes the terms of higher order of Δt.
2. Further assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$, *i.e.,* almost zero.
3. The number of arrivals in non-overlapping intervals are statistically independent, *i.e.,* the process has independent increments.

We now wish to determine the probability of n arrivals in a time interval of length t, denoted by $P_n(t)$. Clearly, n will be an integer greater than or equal to zero. To do so, we shall first develop the differential-difference equations governing the process in two different situations.

Fig. 23.4.

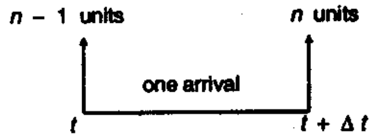

Fig. 23.5.

Case I. ***When n > 0.*** For $n > 0$, there may be two mutually exclusive ways of having n units at time $t + \Delta t$.

(*i*) There are n units in the system at time t and no arrival takes place during time interval Δt. Hence, there will be n units at time $t + \Delta t$ also. This situation is better explained in *Fig. 10.4.*

Therefore, the probability of these two combined events will be

= Prob. of n units at time t × Prob. of no arrival during $\Delta t = P_n(t) \cdot (1 - \lambda\Delta t)$ ...(10.1)

[since prob. of exactly one arrival in $\Delta t = \lambda\Delta t$, prob. of no arrival becomes $= 1 - \lambda\Delta t$.]

(*ii*) *Alternately*, there are $(n - 1)$ units in the system at time t, and one arrival takes place during Δt. Hence there will remain n units in the system at time $t + \Delta t$. This situation is better explained in *Fig. 10.5*.

Therefore, the probability of these two combined events will be

= Prob. of $(n - 1)$ units at time t × Prob. of one arrival in time $\Delta t = P_{n-1}(t).\lambda\Delta t$ ...(10.2)

Note. Since the probability of more than one arrival in Δt is assumed to be negligible, other alternatives do not exist.

Now, adding above two probabilities [given by (10.1) and (10.2)], we get the probability of n arrivals at time $t + \Delta t$, *i.e.*

$$P_n(t + \Delta t) = P_n(t)(1 - \lambda\Delta t) + P_{n-1}(t)\,\lambda\Delta t \qquad \text{...(10.3)}$$

Case 2. *When n = 0.*

$$P_0(t + \Delta t) = \text{Prob. [}\textit{no unit at time } t\text{]} \times \text{Prob. [}\textit{no arrival in time } \Delta t\text{]}$$

$$\therefore \qquad P_0(t + \Delta t) = P_0(t)(1 - \lambda\Delta t). \qquad \text{...(10.4)}$$

Rewriting the equations (10.3) and (10.4) after transposing the terms $P_n(t)$ and $P_0(t)$ to left hand sides, respectively, we get

$$P_n(t + \Delta t) - P_n(t) = P_n(t)(-\lambda\Delta t) + P_{n-1}(t)\,\lambda\Delta t, \qquad n > 0 \qquad \text{...(10.3)}'$$

$$P_0(t + \Delta t) - P_0(t) = P_0(t)(-\lambda\Delta t) \qquad n = 0 \qquad \text{...(10.4)}'$$

Dividing both sides by Δt and then taking limit as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) \qquad \text{...(10.5)}$$

$$\lim_{\Delta t \to 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) \qquad \text{...(10.6)}$$

Since by definition of first derivative, $\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \frac{d\,P_n(t)}{dt} = P_n'(t)$,

the equations (10.6) and (10.5) respectively can be written as

$$P_0'(t) = -\lambda P_0(t), \qquad n = 0 \qquad \text{...(10.7)}$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n > 0 \qquad \text{...(10.8)}$$

This is known as the *system of differential-difference equations.*

To solve the equations (10.7) and (10.8) by iterative method :

Equation (10.7) can be written as

$$\frac{P_0'(t)}{P_0(t)} = -\lambda \quad \text{or} \quad \frac{d}{dt}[\log P_0(t)] = -\lambda \qquad \text{...(10.9)}$$

Integrating both sides w.r.t. 't',

$$\log P_0(t) = -\lambda t + A \qquad \text{...(10.10)}$$

The constant of integration can be determined by using the boundary conditions :

$$P_n(0) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Substituting $t = 0$, $P_0(0) = 1$ in (10.10), find $A = 0$. Thus, (10.10) gives

$$\log P_0(t) = -\lambda t \quad \text{or} \quad P_0(t) = e^{-\lambda t} \qquad \text{...(10.11)}$$

Putting $n = 1$ in (10.8),

$$P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$$

or

$$P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}. \qquad \text{...(10.12)}$$

Since this is the linear differential equation of first order, it can be easily solved by multiplying both sides of this equation by the integrating factor, I.F. $= e^{\int \lambda dt} = e^{\lambda t}$.

Thus, eqn. (10.12) becomes

$$e^{\lambda t}[P_1'(t) + \lambda P_1(t)] = \lambda \quad \text{or} \quad \frac{d}{dt}[e^{\lambda t} P_1(t)] = \lambda$$

Now integrating both sides w.r.t. 't'

$$e^{\lambda t} P_1(t) = \lambda t + B, \quad \text{...(10.13)}$$

where B is the constant of integration.

In order to determine the constant B, put $t = 0$ in (10.13), and get

$$P_1(0) = 0 + B \text{ or } B = 0 \qquad [\because P_1(0) = 0]$$

Substituting $B = 0$ in (10.13), $P_1(t) = \dfrac{\lambda t\, e^{-\lambda t}}{1\,!}$ ...(10.14)

Similarly, putting $n = 2$ in (10.8) and using the result (10.14), we get the equation

$$P_2'(t) + \lambda P_2(t) = \lambda \frac{(\lambda t)\, e^{-\lambda t}}{1\,!} \quad \text{or} \quad \frac{d}{dt}[e^{\lambda t} P_2(t)] = \frac{\lambda\,(\lambda t)}{1\,!}.$$

Integrating w.r.t. 't' $\quad e^{\lambda t} P_2(t) = \dfrac{(\lambda t)^2}{2\,!} + C,$

Put $t = 0$, $P_2(0) = 0$ to obtain $C = 0$. Hence

$$P_2(t) = \frac{(\lambda t)^2\, e^{-\lambda t}}{2\,!}, \text{ for } n = 2 \quad \text{...(10.15)}$$

Similarly, obtain $\quad P_3(t) = \dfrac{(\lambda t)^3\, e^{-\lambda t}}{3\,!}$, for $n = 3$

Likewise, in general, $\quad P_m(t) = \dfrac{(\lambda t)^m\, e^{-\lambda t}}{m\,!}$ for $n = m$. ...(10.16)

If, anyhow, it can be proved that the result (10.16) is also true for $n = m + 1$, then by induction hypothesis result (10.16) will be true for general value of n.

To do so, put $n = m + 1$ in (10.8) and get

$$P'_{m+1}(t) + \lambda P_{m+1}(t) = \lambda \frac{(\lambda t)^m\, e^{-\lambda t}}{m\,!} \qquad \text{[using the results (10.16)]}$$

or

$$\frac{d}{dt}[e^{\lambda t} P_{m+1}(t)] = \frac{(\lambda t)^m\,(\lambda)}{m\,!}.$$

Integrating both sides, $\quad e^{\lambda t} P_{m+1}(t) = \dfrac{(\lambda t)^{m+1}}{(m+1)\, m\,!} + D,$

Again, putting $t = 0$, $P_{m+1}(0) = 0$, we get $D = 0$. Therefore,

$$\therefore \quad P_{m+1}(t) = \frac{(\lambda t)^{m+1}\, e^{-\lambda t}}{(m+1)\,!}.$$

Hence, in general, $\quad P_n(t) = \dfrac{(\lambda t)^n\, e^{-\lambda t}}{n\,!}$, ...(10.17)

which is a ***Poisson distribution formula.*** This completes the proof of the theorem.

Note. After carefully understanding the above procedure, the students can much reduce the number of steps by solving the differential equation of the standard-from : $y' + P(x)\,y = Q(x)$, using the formula

$$y \,.\, e^{\int P dx} = \int Q(x)\,(e^{\int P dx})\, dx + C,$$

where $e^{\int P dx}$ is the integrating factor (I.F.)

Alternative Method : *Generating Function Technique.*

The system of equations (10.7) and (10.8) is

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n > 0 \quad \text{...(i)}$$

$$P_0'(t) = -\lambda P_0(t), \qquad n = 0 \quad \text{...(ii)}$$

We define the generating function of $P_n(t)$ as, $P(z, t) = \sum_{n=0}^{\infty} P_n(t)\, z^n$. Also, $P'(z, t) = \sum_{n=0}^{\infty} P_n'(t)\, z^n$.

Multiplying both sides of (i) by z^n and taking summation for $n = 1, 2, ..., \infty$, we get

$$\sum_{n=1}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=1}^{\infty} z^n P_n(t) + \lambda \sum_{n=1}^{\infty} P_{n-1}(t)\, z^n \qquad \text{...(iii)}$$

Now adding (ii) and (iii), we get

$$\sum_{n=0}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=0}^{\infty} z^n P_n(t) + \lambda \sum_{n=0}^{\infty} z^{n+1} P_n(t)$$

or $$P'(z, t) = -\lambda P(z, t) + \lambda z P(z, t) \quad \text{or} \quad \frac{P'(z, t)}{P(z, t)} = \lambda (z-1)$$

or $$\frac{d}{dt}[\log P(z, t)] = \lambda (z-1)$$

Integrating both sides, $\log P(z, t) = \lambda (z-1)\, t + E.$ ...(iv)

To determine E, we put $t = 0$ to get $\log P(z, 0) = E$

But, $P(z, 0) = \sum_{n=0}^{\infty} z^n P_n(0) = P_0(0) + \sum_{n=1}^{\infty} z^n P_n(0)$

$= 1 + 0 = 1$ ($\because P_0(0) = 1$, and $P_n(0) = 0$ for $n > 0$)

Therefore, $E = \log P(z, 0) = \log 1 = 0.$

$\therefore$ eqn. (iv) becomes, $\log P(z, t) = \lambda(z-1)\,t$ or $P(z, t) = e^{\lambda (z-1) t}$.

Now, $P_n(t)$ can be defined as $P_n(t) = \frac{1}{n!}\left[\frac{d^n P(z, t)}{dz^n}\right]_{z=0}$

Using this formula, $P_0(t) = [P(z, t)]_{z=0} = e^{-\lambda t}$

$$P_1(t) = \left[\frac{d P(z, t)}{dz}\right]_{z=0} = [e^{\lambda (z-1) t} \lambda t]_{z=0} = \frac{e^{-\lambda t} \lambda t}{1!}$$

$$P_2(t) = \frac{1}{2!}\left[\frac{d^2 P(z, t)}{dz^2}\right]_{z=0} = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

... ...

In general, $$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \qquad \text{...(v)}$$

Thus the probability of n arrivals in time 't' follows the ***Poisson law*** given by eqn. (v).

Q. 1. Show that 'n' the number of arrivals in a queue in time t follows the Poisson distribution, stating the assumptions clearly.
2. Show that the distribution of the number of births up to time 'T' in a simple birth process follows the Poisson law.
3. What do you understand by a queue ? Give some applications of queueing theory.
4. Explain what do you mean by Poisson process. Derive the Poisson distribution, given that the probability of single arrival during a small time interval Δt is $\lambda \Delta t$ and that of more than one arrival is negligible. **[JNTU (B. Tech.) 2002; Meerut (Maths.) 96]**
5. State when a model is called Pure Birth Process in Queueing Theory. **[Bhubneshwar (IT) 2004]**

10.7-2. Properties of Poisson Process of Arrivals

It has already been derived that–*if n be the number of arrivals during time interval t, then the law of probability in Poisson process is given by*

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, ..., \infty \qquad \text{...(10.18)}$$

where λt is the parameter.

(1) Since mean $E(n) = \lambda t$, and var. $(n) = \lambda t$, ...(10.19)

the average (expected) number of arrivals in unit time will be

$E(n)/t = \lambda =$ mean arrival rate (or input rate).

(2) If we consider the time interval $(t, t + \Delta t)$, where Δt is sufficiently small, then

$P_0(\Delta t) =$ Prob [*no arrival in time* Δt]

Putting $n = 0$ and $t = \Delta t$ in (10.18)

$$P_0(\Delta t) = \frac{e^{-\lambda\Delta t}}{0\,!} = e^{-\lambda\Delta t} = 1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2\,!} - \ldots = 1 - \lambda\Delta t + O(\Delta t)$$

where the term $O(\Delta t)$ indicates a quantity that is negligible compared to Δt. More precisely, $O(\Delta t)$ represents any function of Δt such that

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0.$$

[For example, $(\Delta t)^2$ can be replaced by $O(\Delta t)$ because $\lim_{\Delta t \to 0} \frac{(\Delta t)^2}{\Delta t} = 0$. This notation will be very useful for summarizing the negligible terms which do not enter in the final result]

$$\therefore \quad P_0(\Delta t) = 1 - \lambda\Delta t \qquad \text{...(10.20)}$$

which means that the probability of no arrival in Δt is $1 - \lambda\Delta t$. In the similar fashion, $P_1(\Delta t)$ can be written as

$$P_1(\Delta t) = \frac{(\lambda\Delta t)\, e^{-\lambda\Delta t}}{1\,!} \qquad [\text{putting } n = 1,\ t = \Delta t \text{ in (10.18)}]$$

$$= \lambda\Delta t \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2\,!} - \ldots \right] \qquad = \lambda\Delta t + O(\Delta t).$$

Neglecting the term $O(\Delta t)$, $\quad P_1(\Delta t) = \lambda\Delta t$, ...(10.21)

which means that the probability of one arrival in time Δt is $\lambda\Delta t$.

Similarly, $\quad P_2(\Delta t) = \dfrac{(\lambda\Delta t)^2\, e^{-\lambda\Delta t}}{2\,!} = (\lambda\Delta t)^2 \left[1 - \lambda\Delta t + \dfrac{(\lambda\Delta t)^2}{2\,!} - \ldots \right] = O(\Delta t).$

Again neglecting the term $O(\Delta t)$, we have $\quad P_2(\Delta t) = 0$, ...(10.22)

ans so on. Thus, it is concluded from the property of Poisson process that the probability of more than one arrival in time Δt is negligibly small, provided the terms of second and higher order of Δt are considered to be negligibly small. Symbolically,

$$P_n(\Delta t) = \text{ negligibly small for all } n > 1. \qquad \text{...(10.23)}$$

Q. State some important properties of Poission's process. **[JNTU (B. Tech.) 2003]**

10.7-3. Distribution of Inter-Arrival Times (Exponential Process)

Let T be the time between two consecutive arrivals (called the inter-arrival time), and $a(T)$ denotes the probability density function of T. Then the following important theorem can be proved.

Theorem 10.2. *If n, the number of arrivals in time t, follows the Poisson distribution,*

$$P_n(t) = (\lambda t)^n\, e^{-\lambda t}/n\,!, \qquad \text{...(10.24)}$$

then T (the inter-arrival time) obeys the negative exponential law

$$a(T) = \lambda e^{-\lambda T} \qquad \text{...(10.25)}$$

and vice-versa. **[Kanpur 2000; Garhwal M.Sc. (Stat.) 95; Raj .Univ. (M.Phil) 91]**

Proof. Suppose that $t_0 =$ instant of an arrival initially.

Since there is no arrival in the intervals $(t_0, t_0 + T)$ and $(t_0 + T, t_0 + T + \Delta T)$, therefore $(t_0 + T + \Delta T)$ will be the instant of subsequent arrival.

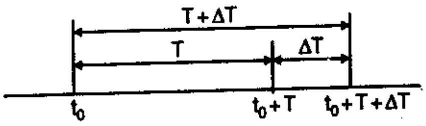

Fig. 10.6

Therefore, putting $t = T + \Delta T$ and $n = 0$ in (5.24),

$$P_0(T + \Delta T) = \frac{[\lambda(T + \Delta T)]^0 \,.\, e^{-\lambda(T + \Delta T)}}{0\,!} = e^{-\lambda(T + \Delta T)}$$

$$= e^{-\lambda T} \,.\, e^{-\lambda\Delta T} = e^{-\lambda T}\ [1 - \lambda\Delta T + O(\Delta T)]$$

Since $\quad P_0(T) = e^{-\lambda T}$ from (10.24),

$$P_0(T + \Delta T) = P_0(T)\ [1 - \lambda\Delta T + O(\Delta T)]$$

or $\quad P_0(T + \Delta T) - P_0(T) = P_0(T)\ [-\lambda\Delta T + O(\Delta T)].$

Dividing both sides by ΔT,

$$\frac{P_0(T+\Delta T)-P_0(T)}{\Delta T} = -\lambda P_0(T) + \frac{O(\Delta T)}{\Delta T} P_0(T)$$

Now taking limit on both sides as $\Delta T \to 0$,

$$\lim_{\Delta T \to 0} \frac{P_0(T+\Delta T)-P_0(T)}{\Delta T} = \lim_{\Delta T \to 0} \left[-\lambda P_0(T) + \frac{O(\Delta T)}{\Delta T} P_0(T) \right]$$

or
$$\frac{dP_0(T)}{dT} = -\lambda P_0(T). \left[\text{since} \lim_{\Delta T \to 0} \frac{O(\Delta T)}{\Delta T} = 0 \right] \quad ...(10.26)$$

But, L.H.S. of (10.26) is denoting the *probability density function* of T, say $a(T)$. Therefore,

$$*a(T) = **\lambda P_0(T). \quad \textbf{(see footnote)} \quad ...(10.27)$$

But, from equation (10.24), $P_0(T) = e^{-\lambda T}$. Putting this value of $P_0(T)$ in (10.27),

$$a(T) = \lambda e^{-\lambda T} \quad ...(10.28)$$

which is the *exponential law of probability* for T with mean $1/\lambda$ and variance $1/\lambda^2$, *i.e.*,

$$E(T) = 1/\lambda, \text{ Var. } (T) = 1/\lambda^2.$$

In a similar fashion, the converse of this theorem can be proved.

Q. 1. Give the axioms characterizing a Poisson process. If the number of arrivals in some time interval follows a Poisson distribution, show that the distribution of the time interval between two consecutive arrivals is exponential.
[Delhi M.A/M.Sc. (Stat.) 95; Raj. Univ. (M. Phil) 91]

2. Show that if the inter-arrival times are negative exponentially distributed, the number of arrivals in a time period is a Poisson process and conversely.

3. If the intervals between successive arrivals are i.i.d. random variables which follow the negative exponential distribution with mean $1/\lambda$, then show that the arrivals form a Poisson Process with mean λt. **[Garhwal M.Sc. (Stat.) 91]**

4. Show that inter-arrival times are distributed exponentially, if arrival is a Poisson process. Prove the converse also.
[Delhi M.A/M.Sc (OR) 92.]

5. State the three axioms underlying the exponential process. Under exponential assumptions can two events occur during a very small interval. **[Meerut 2002]**

10.7-4. Markovian Property of Inter-arrival Times

Statement. *The Markovian property of inter-arrival times states that at any instant the time until the next arrival occurs is independent of the time that has elapsed since the occurrence of the last arrival. That is to say,*

Prob. $[T \geq t_1 \mid T \geq t_0] = $ *Prob.* $[0 \leq T \leq t_1 - t_0]$

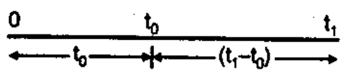

Fig. 10.7

Proof. Consider

$$\text{Prob. } [T \geq t_1 \mid T \geq t_0] = \frac{\text{Prob. } [(T \geq t_1) \text{ and } (T \geq t_0)]}{\text{Prob. } [T \geq t_0]} \quad \text{(formula of conditional probability)} \quad ...(10.29)$$

Since the inter-arrival times are exponentially distributed, the right hand side of equation (10.29) can be written as

$$\frac{\int_{t_0}^{t_1} \lambda e^{-\lambda t}\, dt}{\int_{t_0}^{\infty} \lambda e^{-\lambda t}\, dt} = \frac{e^{-\lambda t_1} - e^{-\lambda t_0}}{-e^{-\lambda t_0}}$$

$$\therefore \quad \text{Prob. } [T \geq t_1 \mid T \geq t_0] = 1 - e^{-\lambda (t_1 - t_0)} \quad ...(10.30)$$

---

* According to probability distributions $d/dx\,[F(x)] = f(x)$, where $F(x)$ is the *'distribution function'* and $f(x)$ is the *'probability density function'*. Hence by the similar argument, we may write $d/dT\,[P_0(T)] = a(T)$, where $P_0(T)$ is the probability distribution function for no arrival in time T, and $a(T)$ is denoting the corresponding probability density function of T.

** Since *'probability density function'* is always non-negative, so neglect the negative sign from right side of equation (23.26).

But, $$\text{Prob. } [0 \le T \le t_1 - t_0] = \int_0^{t_1 - t_0} \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda (t_1 - t_0)}. \qquad ...(10.31)$$

Thus, by virtue of equations (10.30) and (10.31), it can be concluded that

$$\text{Prob. } [T \ge t_1 \mid T \ge t_0] = \text{Prob. } [0 \le T \le t_1 - t_0].$$

This proves the *Markovian property* of inter-arrival times.

Q. State and prove the Markovian property of inter-arrival times (*i.e.* of exponential distribution).

10.7-5. Distribution of Departures (or Pure Death Process)

In this process assume that there are N customers in the system at time $t = 0$. Also, assume that no arrivals (births) can occur in the system. Departures occur at a rate μ per unit time, *i.e.*, output rate is μ. We wish to derive the distribution of departures from the system on the basis of the following three *axioms* :

(1) Prob. [*one departure during* Δt] $= \mu\Delta t + O(\Delta t)^2 = \mu\Delta t$ [$\because$ $O(\Delta t)^2$ is negligible]

(2) Prob. [*more than one departure during* Δt] $= O(\Delta t)^2 = 0$.

(3) The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable, *i.e.*, the process $N(t)$ has independent increments.

First obtain the differential difference equation in three mutually exclusive ways :

Case I. *When* $0 < n < N$. Proceeding exactly as in the *Pure Birth Process*,

$$P_n(t + \Delta t) = P_n(t)\,[1 - \mu\Delta t] + P_{n+1}(t)\,\mu\Delta t \qquad ...(10.32)$$

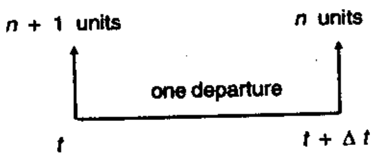

Fig. 10.8

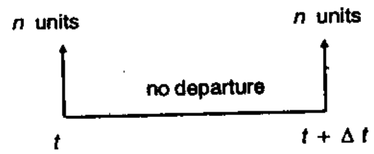

Fig. 10.9

Case II. *When* $n = N$. Since there are exactly N units in the system, $P_{n+1}(t) = 0$,

$$\therefore \quad P_N(t + \Delta t) = P_N(t)\,[1 - \mu\Delta t] \qquad ...(10.33)$$

Case III. *When* $n = 0$.

$$P_0(t + \Delta t) = P_0(t) + P_1(t)\,\mu\Delta t \qquad ...(10.34)$$

Since there is no unit in the system at time t, the question of any departure during Δt does not arise. Therefore, probability of no departure is unity in this case.

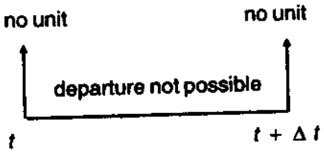

Fig. 10.10.

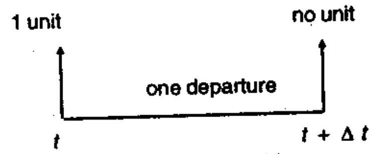

Fig. 10.11.

Now, re-arranging the terms and dividing by Δt, and also taking the limit $\Delta t \to 0$ the equations (10.33), (10.32) and (10.34), respectively, become

$$P_N'(t) = -\mu P_N(t), \qquad n = N \qquad ...(10.35)$$

$$P_n'(t) = -\mu P_n(t) + \mu P_{n+1}(t), \qquad 0 < n < N \qquad ...(10.36)$$

$$P_0'(t) = \mu P_1(t), \qquad n = 0 \qquad ...(10.37)$$

To solve the system of equations (10.35), (10.36) and (10.37) :

Iterative method can be used to solve the system of three equations.

Step 1. From equation (10.35) obtain

$$\frac{P_N'(t)}{P_N(t)} = -\mu \quad \text{or} \quad \frac{d}{dt} \log P_N(t) = -\mu.$$

Integrating both sides of this equation,

$$\log P_N(t) = -\mu t + A \quad \text{...(10.38)}$$

To determine 'A', use the boundary condition $P_N(0) = 1$, and thus get $A = 0$ ($\because \log 1 = 0$).

Therefore, equation (10.38) becomes

$$\log P_N(t) = -\mu t \text{ or } P_N(t) = e^{-\mu t} \quad \text{...(10.39)}$$

Step 2. In equation (10.36), put $n = N - 1$, and the value of $P_N(t)$ from equation (10.39),

$$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu P_N(t)$$

or $$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu e^{-\mu t} \quad \text{[from equation (10.39)]}$$

or $$P'_{N-1}(t) + \mu P_{N-1}(t) = \mu e^{-\mu t} \quad \text{...(10.40)}$$

The solution of this equation is given by

$$P_{N-1}(t)\, e^{\mu t} = \int \mu e^{-\mu t} e^{\mu t}\, dt + B \quad (\because \text{I.F.} = e^{\mu t})$$

or $$P_{N-1}(t) = \mu t\, e^{-\mu t} + Be^{-\mu t} \quad \text{...(10.41)}$$

To determine B, put $t = 0$, $P_{N-1}(t) = 0$ in (10.41) and get $B = 0$. Therefore,

$$P_{N-1}(t) = \frac{\mu t\, e^{-\mu t}}{1\,!}$$

Step 3. Putting $n = N - 2$ in equation (10.36) and proceeding exactly as in *Step 2*,

$$P_{N-2}(t) = \frac{e^{-\mu t} (\mu t)^2}{2\,!}$$

Step 4. Now, putting $n = N - 3, N - 4, \ldots, N - i$, and using induction process

$$P_{N-3}(t) = \frac{e^{-\mu t} (\mu t)^3}{3\,!}$$

$$\ldots \quad \ldots$$

$$P_{N-i}(t) = \frac{e^{-\mu t} (\mu t)^i}{i\,!}, \quad i = 0, 1, 2, \ldots, N-1$$

$$\ldots \quad \ldots$$

In general, on letting $n = N - i$ $$P_n(t) = \frac{e^{-\mu t} (\mu t)^{N-n}}{(N-n)\,!}, \quad n = 1, 2, \ldots, N \quad \text{...(10.42)}$$

Step 5. In order to find $P_0(t)$, use the following procedure,

Since $$1 = \sum_{n=0}^{N} P_n(t) = P_0(t) + \sum_{n=1}^{N} P_n(t)$$

$\therefore$ $$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t) = 1 - \sum_{n=1}^{N} \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!} \quad \text{...(10.43)}$$

Finally, combining the results (10.42) and (10.43)

$$P_n(t) = \begin{cases} \dfrac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!}, & \text{for } n = 1, 2, \ldots, N \\ 1 - \displaystyle\sum_{n=1}^{N} \dfrac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)\,!}, & \text{for } n = 0 \end{cases} \quad \text{...(10.44)}$$

Thus, the number of departures in time t follows the *'Truncated Poisson Distribution'*.

Q. Establish the probability distribution formula for Pure-Death Process.

10.7-6. Derivation of Service Time Distribution

Let T be the random variable denoting the service time and t the possible value of T.

Let $S(t)$ and $s(t)$ be the *cumulative density function* and the *probability density function* of T, respectively.

To find $s(t)$ for the Poisson departure case, it has been observed that the probability of no service during time 0 to t is equivalent to the probability of having no departure during the same period.

Thus, Prob. [*service time* $T \geq t$] = Prob. [*no departure during* t] = $P_N(t)$

where there are N units in the system and no arrival is allowed after N. Therefore, $P_N(t) = e^{-\mu t}$

$$S(t) = \text{Prob.}\ (T \leq t) = 1 - \text{Prob.}\ [T \geq t] \quad \text{or} \quad S(t) = 1 - e^{-\mu t}$$

Differentiating both sides, w.r.t. 't', we get

$$\frac{d}{dt} S(t) = s(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Thus, it is concluded that the service time distribution is '*Exponential*' with mean $1/\mu$ and variance $1/\mu^2$.

Thus, mean service time = $1/\mu$.

Q. Explain the role of exponential distribution and its characteristics. **[Bhubneshwar (IT) 2004]**

10.7-7. Analogy of Exponential Service Times with Poisson Arrivals

It has been proved in **sec. 10.7-3** that if number of arrivals (n) follows the *Poisson distribution*, then the inter-arrival time (T) will follow the *exponential one*, and *vice-versa.*

In the like manner, it can also be shown that, if the time (t) to complete the service of a unit follows the *exponential distribution* given by the probability density function

$$s(t) = \mu e^{-\mu t} \qquad \text{...(10.45)}$$

where μ is the mean servicing rate for a particular station, then the number (n) of departures in time T (if there were no enforced idle time) will follow the *Poisson distribution* given by

$$\varphi_T(n) = \text{Prob. [}n \text{ services in time } T \text{, if servicing is going on throughout } T] = (\mu T)^n e^{-\mu T}/n\,! \qquad \text{...(10.46)}$$

Consequently, from (10.27), it can be shown that

$$\varphi_{\Delta T}(0) = \text{Prob. [}\textit{no service in } \Delta T] = 1 - \mu \Delta T \qquad \text{...(10.47)}$$

and

$$\varphi_{\Delta T}(1) = \text{Prob. [}\textit{one service in } \Delta T] = \mu \Delta T. \qquad \text{...(10.48)}$$

10.7-8. Erlang Service Time Distribution (E_k).

So far it is considered (in **10.7-3** and **10.7-6**) and seen that the inter-arrival time distribution and service time distribution both will follow the exponential assumptions given by

$$a(T) = \lambda e^{-\lambda t}, \text{ and } s(t) = \mu e^{-\mu t}, \text{ respectively.} \qquad \text{...(10.49)}$$

These only give a one particular family of possible arrival and service time distribution, respectively.

A two parameter (μ and k) generalisation of the exponential family, which is of great importance in queueing problems is called the *Erlang* family of service time distribution (named for *A.K. Erlang*, the ***Danish*** telephone engineer. This is defined by its probability density function,

$$s(t, \mu, k) = (k\mu)^k t^{k-1} e^{-k\mu t}/(k-1)\,! = C_k t^{k-1} e^{-k\mu t} \qquad \text{...(10.50)}$$

where $C_k = (k\mu)^k/(k-1)\,!$, $0 \leq t < \infty$, $k \geq 1$.

It should be noted carefully that (10.50) gives us the exponential distribution given by (10.49) for $k = 1$.

Let $t_1, t_2, t_3, ..., t_k$ be the servicing time for any customer in respective k phases, then the total service time t is given by

$$t = t_1 + t_2 + \ldots + t_k.$$

Also, each of the times $t_1, t_2, ..., t_k$ is independently and exponentially distributed with parameter $k.\mu$.

Hence, $P\,[t \leq t_1 + t_2 \ldots + t_k \leq t + \Delta t]$

$$= \iint \ldots \int p(t_1)\, p(t_2) \ldots p(t_k)\, dt_1\, dt_2 \ldots dt_k. \text{ for } t \leq t_i \leq t + \Delta t,\ i = 1, 2, ..., k.$$

$$= \iint \ldots \int (k\,\mu e^{-k\mu t_1} \ldots (k\,\mu e^{-k\mu t_k})\, dt_1 \ldots dt_k. \quad [\text{since } p(t_i) = k\,\mu e^{-k\mu t_i}]$$

$$= (k\,\mu)^k \iint \ldots \int e^{-k\mu \sum_{i=1}^{k} t_i}\, dt_1 \ldots dt_k.$$

Now applying ***Drichlet's*** theorem of multiple integrals,

$$= (k\,\mu)^k \frac{\Gamma 1^k}{\Gamma(k)} e^{-k\mu t}\, t^{k-1}$$

$$= \frac{(k\mu)^k}{\Gamma(k)} t^{k-1} e^{-k\mu t}, k \geq 0.$$

Note. The distribution is a modified χ^2 distribution with mean $(1/\mu)$ and $2k$ degrees of freedom.

Thus, if we have service times $t_1, t_2, t_3, ..., t_k$ in k phases which are exponentially distributed variables with a common mean $1/k\mu$, then $t = t_1 + t_2 + t_3 + ... + t_k$ has the *Erlangian (Gamma)* distribution with k phases and parameter μ.

Derivation of Erlangion Service Distribution

In the *Fig.* 10.12 for *Erlangian service time distribution* for $k = 3$ phases, it is observed that—

(*i*) each phase of service is exponential;

(*ii*) a unit enters phase 3 first, then goes to 2, to 1 and out;

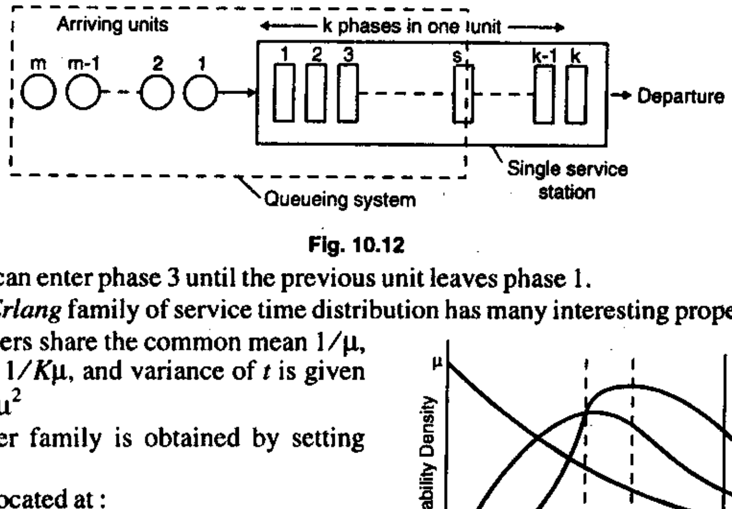

Fig. 10.12

(*iii*) no other unit can enter phase 3 until the previous unit leaves phase 1.

Properties. The *Erlang* family of service time distribution has many interesting properties such as :

(1) All the members share the common mean $1/\mu$, that is, $E(t) = 1/K\mu$, and variance of t is given by $V(t) = 1/k\mu^2$

(2) One parameter family is obtained by setting $k = 1$.

(3) The mode is located at :

$t = 0$	for $k = 1$,
$t = 1/2\mu$	for $k = 2$,
---	---
$t = (k-1)/k\mu$	for general k,
$t = 1/\mu$	for $k \to \infty$

(4) As $k \to \infty$, $V(t) \to 0$, [since $V(t) = 1/k\mu^2$]

(5) For constant service time, $k \to \infty$ **[Note.]**

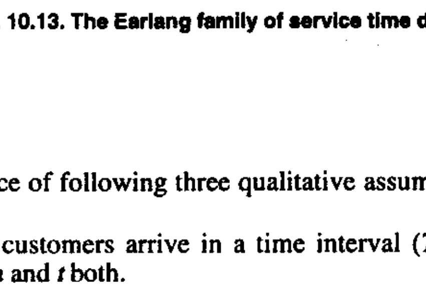

Fig. 10.13. The Earlang family of service time distribution.

10.8. SOME QUALITATIVE ASSUMPTIONS

It is essential to pay much attention to physical significance of following three qualitative assumptions for further discussion of queueing system.

(a) **Stationary waiting line.** The probability that n customers arrive in a time interval $(T, T + t)$ is independent of T and is a function of the variables n and t both.

(b) **Absence of after effects.** This means that the probability of n customers arriving during a time interval $(T, T + t)$ does not depend on the number of customers arriving before T.

(c) **The orderlines of the waiting line.** It expresses the practical impossibility of two or more customers arriving at the same instant of time.

A waiting line (queue) satisfying the above three conditions is usually called a *Simple Queue*.

Q. 1. (*a*) Explain the basic queueing process. What are the important random variates in queueing system to be investigated ?

(*b*) What do you understand by (*i*) queue discipline (*ii*) input, and (*iii*) holding time ?

2. Explain : (*i*) the constituents of a queueing model, and (*ii*) the characteristics to be analysed.

3. State some of the important distributions of arrival intervals and service times. [Delhi MA/M.Sc (OR) 93]
4. Give the essential characteristics of the queueing process.
5. State some of the important inter-arrival and service time distributions.
6. What do you understand by a queue ? Give some important applications of queueing theory.
7. Find the mode of the k-Erlang distribution with parameter μ.
8. What do you understand by a optimum service rate ? Show how some important queueing formulae are used in determining the optimum service rate and the number of channels.
9. Prove that, for the Erlang distribution with parameters μ and k, the mode is at $(1 - 1/k)\, 1/\mu$, the mean is $1/\mu$, and the variance is $1/k\mu^2$. [Kanpur 96; Garhwal M.Sc. (Stat) 92; Meerut (M.Sc. Maths) 90]
10. Write a note on *Erlangian* distribution. [Meerut (OR) 2003; Delhi M.A/M.Sc. (OR.) 90]

10.9. KENDALL'S NOTATION FOR REPRESENTING QUEUEING MODELS

Generally queueing model may be completely specified in the following symbolic form : $(a \mid b \mid c) : (d \mid e)$, where

$a \equiv$ probability law for the arrival (or inter-arrival) time.
$b \equiv$ probability law according to which the customers are being served,
$c \equiv$ number of channels (or service stations),
$d \equiv$ capacity of the system, *i.e.*, the maximum number allowed in the system (in service and waiting),
$e \equiv$ queue discipline.

It is important to note that first three characteristics $(a \mid b \mid c)$ in the above notation were introduced by *D. Kendall* (1953). Later, *A. Lee* (1966) added the *fourth* (d) and the *fifth* (e) characteristics to the notation. Although, it is noticed that this notation is not suitable for describing complex models such as queues in series, or network queues. This will be suitable, however, for the purpose of material presented here, and the reader should find it helpful in comparing the different models.

Q. Explain Kendall's notations for representing queueing models. [JNTU (Mech. & Prod.) 2004]

10.10. CLASSIFICATION OF QUEUEING MODELS

For simplicity, the queueing models can be basically classified as follows :

I—Probabilistic Queueing Models

Model I. (***Erlang Model***). This model is symbolically represented by $(M \mid M \mid 1) : (\infty \mid FCFS)$. This denotes *Poisson* arrival (exponential inter-arrival), *Poisson* departure (exponential service time), single server, infinite capacity and *"First Come, First Served"* service discipline.

Note. Since the *'Poisson'* and the *'Exponential'* distributions are related to each other (see **sec. 10.7-3**), both of them are denoted by the same letter '*M*'. Letter '*M*' is used due to *Markovian property* of exponential process.

Model II. (***General Erlang Model***). Although this model is also represented by $(M \mid M \mid 1) : (\infty \mid FCFS)$, but this is a general queueing model in which the rate of arrival and the service depend on the length n of the line.

Model III. This model is represented by $(M \mid M \mid 1) : (N \mid FCFS)$. In this model, capacity of the system is limited (finite), say N. Obviously, the number of arrivals will not exceed the number N in any case.

Model IV. This model is represented by $(M \mid M \mid s) : (\infty \mid FCFS)$, in which the number of service stations is s in parallel.

Model V. This model is represented by $(M \mid E_k \mid 1) : (\infty \mid FCFS)$, that is, *Poisson* arrivals, *Erlangian* service time for k phases and a single server.

Model VI. (***Machine Repairing Model***). This model is represented by $(M \mid M \mid R) : (K \mid GD)$, $K > R$, that is, Poisson arrivals, Exponential service time, R repairmen, and K machines in the system, and general queue discipline.

Model VII. ***Power-Supply Model.***

Model VIII. ***Economic Cost Profit Models.***

Model IX. $(M \mid G \mid 1) : (\infty \mid GD)$, where G is the general output distribution, and GD represents a general service discipline.

II—Mixed Queueing Model

Model X. $(M \mid D \mid 1) : (\infty \mid FCFS)$, where D stands for deterministic service time.

III—Deterministic Queueing Model

Model XI. $(D \mid D \mid 1) : (K - 1 \mid FCFS)$, where

$D \rightarrow$ Deterministic arrivals, *i.e.*, inter-arrival time distribution is *constant* or *regular*.
$D \rightarrow$ Deterministic service time distribution.

Q. 1. Give a brief summery of the various types of queueing models. **[Bhubnashwar (IT) 2004]**
2. Write a note on *Kendal* and *Lee's* notation for the identification of queues. **[Karnataka 94]**

10.11. SOLUTION OF QUEUEING MODELS AND LIMITATIONS FOR ITS APPLICATIONS

The solution of queueing models as classified in sec. **10.10** may consist of the following parts :

(*a*) ***To obtain the system of steady state equations governing the queue.***
(*b*) ***To solve these equations for finding out the probability distribution of queue length.***
(*c*) ***To obtain probability density function for waiting time distribution.***
(*d*) ***To find the busy period distribution.***
(*e*) ***To derive formula for*** $L_s, L_q, (L \mid L > 0), W_s, W_q, (W \mid W > 0)$, ***and*** $Var\{n\}$, ***etc.***
(*f*) ***Also, to obtain the probability of arrival during the service time of any customer.***

Limitation for Application of Queueing Model :

The single channel queueing model can be fitted in situations where the following conditions are satisfied.

(i) The number of arrivals rate is denoted by λ.
(ii) The service time has exponential distribution. The average service rate is denoted by μ.
(iii) Arrivals are from infinite population.
(iv) The queue discipline is *FCFS (i.e. FIFO), i.e.* the customers are served on a first come first served basis—
(v) There is only a single service station.
(vi) The mean arrival rate is less than the mean service rate, *i.e.* $\lambda < \mu$.
(vii) The waiting space available for customers in the queue is infinite.

The single channel queueing model is the most simple model which is based on the above mentioned assumption. But, in reality, there are several limitations of this modes in its applications. One abvious limitation is the possibility that the waiting space, in fact, be limited. Other possibility is that arrival rate is state dependent. That is, potential customers are discouraged from entering the queue if they observe a long line at the time they arrive. Another practical limitation of the model is that the arrival process is not stationary. It is quite possible that the service station would experience peak period, and slack periods during which the arrival rate is higher and lower respectively than the over all average. These could occur at particular times during a day or a week or particular weeks during a year. The population of customers served may be finite, the queue discipline may not be *first come first served.* In general, the validity of these models depends on the assumptions that are often unrealistic in practice.

Even when the assumptions are realistic, there is another limitation of queueing theory that is often overlooked. Queueing models give steady state solution, *i.e.* the model tells us what will happen after queueing system hes been in operation long enough to eliminate the effects of starting with an empty queue at the beginning of each business day. In some applications, the queueing system never reaches a steady state, so the model solutions are of little importance.

Q. 1. Mention any seven conditions that must be fulfilled by the situations if they were to be described by a queueing model. What are the limitations of this model in its applications. **[C.A. (Nov.) 93]**
2. (*a*) Describe Queueing model and its significance. What are various queue models, give in details ?
(*b*) List the factors that constitute the basic elements of queuing model. For each of those, enumerate the alternatives possible. Represent them diagramatically to cover all possible implimentations of a queueing model ? **[IGNOU 99, 98, 96]**

We now proceed to discuss each model in detail with the help of various interesting examples.

10.12. MODEL I. $(M \mid M \mid 1) : (\infty \mid FCFS)$: BIRTH AND DEATH MODEL

This model is also called the ***'birth and death model'***.

I. To obtain the system of steady-state equations. **[Agra 99; Meerut 93]**

The probability that there will be n units $(n > 0)$ in the system at time $(t + \Delta t)$ may be expressed as the sum of three independent compound probabilities, by using the fundamental properties of probability, Poisson arrivals, and of exponential service times.

(*i*) ***The product of three probabilities*** (see **Fig. 10.14**),

(*a*) that there are n units in the system at time $t = P_n(t)$

(*b*) that there is no arrival in time $\Delta t = P_0(\Delta t) = 1 - \lambda \Delta t$ [see (10.20)]

(*c*) that there is no service in time $\Delta t = \varphi_{\Delta t}(0) = 1 - \mu \Delta t$; [see (10.47)]

is given by

$$P_n(t).(1 - \lambda \Delta t).(1 - \mu \Delta t) \cong P_n(t)\,[1 - (\lambda + \mu)\,\Delta t] + O_1(\Delta t).$$

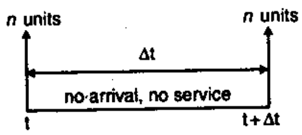

Fig. 10.14.

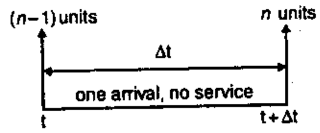

Fig. 10.15

(*ii*) ***The product of three probabilities*** (see **Fig. 10.15**),

(*a*) that there are $(n - 1)$ units in the system at time $t = P_{n-1}(t)$;

(*b*) that there is one arrival in time $\Delta t = P_1(\Delta t) = \lambda \Delta t$ [see (10.21)]

(*c*) that there is no service in $\Delta t = \varphi_{\Delta t}(0) = 1 - \mu \Delta t$;

is given by $\qquad P_{n-1}(t).(\lambda \Delta t).(1 - \mu \Delta t) \cong \lambda P_{n-1}(t)\,\Delta t + O_2(\Delta t)$,

(*iii*) ***The product of probabilities*** (see **Fig. 10.16**),

(*a*) that there are $(n + 1)$ units in the system at time $t = P_{n+1}(t)$

(*b*) that there is no arrival in time Δt

$= P_0(\Delta t) = 1 - \lambda \Delta t$

(*c*) that there is one service in time

$\Delta t = \varphi_{\Delta t}(1) = \mu \Delta t$; [see (10.48)]

is given by

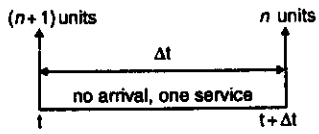

Fig. 10.16

$$P_{n+1}(t)\,(1 - \lambda \Delta t)\,\mu \Delta t \cong P_{n+1}(t)\,\mu \Delta t + O_3(\Delta t).$$

Note. The probabilities of more than one unit arriving and/or being served during the interval Δt are assumed to be negligible. Further, $O_1(\Delta t)$, $O_2(\Delta t)$, $O_3(\Delta t)$ are also the functions of Δt in the sense of notation '$O(\Delta t)$' as explained in sec. 10.7-2.

Now, by adding above three independent compound probabilities, we obtain the probability of n units in the system at time $(t + \Delta t)$, *i.e.*,

$$P_n(t + \Delta t) = P_n(t)\,[1 - (\lambda + \mu)\,\Delta t] + P_{n-1}(t)\,\lambda \Delta t + P_{n+1}(t)\,\mu \Delta t + O(\Delta t), \qquad \ldots(10.51)$$

where $O(\Delta t) = O_1(\Delta t) + O_2(\Delta t) + O_3(\Delta t)$.

The equation (10.51) may be written as

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -(\lambda + \mu)\,P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}$$

Now, taking limit as $\Delta t \to 0$ on both sides,

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t \to 0}\left[-(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{O(\Delta t)}{\Delta t}\right]$$

or
$$\frac{dP_n(t)}{dt} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t);\ n>0 \left(\text{since} \lim_{\Delta t\to 0}\frac{O(\Delta t)}{\Delta t} = 0\right) \quad ...(10.52)$$

In a similar fashion, the probability that there will be no unit (*i.e.* $n = 0$) in the system at time $(t + \Delta t)$ will be the sum of the following two independent probabilities :

(*i*) Prob. [*that there is no unit in the system at time t and no arrival in time* Δt] $= P_0(t).(1 - \lambda\Delta t)$.
question of any service in time Δt does not arise because there are *no* units in the system at time *t;* and

(*ii*) Prob. [*that there is one unit in the system at time t, one unit serviced in* Δt, *and no arrival in* Δt]

$$= P_1(t).\mu\Delta t.(1 - \lambda\Delta t) \cong P_1(t)\mu\Delta t + O(\Delta t)$$

Now, adding these two probabilities, we get

$$P_0(t+\Delta t) = P_0(t)\,[1 - \lambda\Delta t] + P_1(t)\,\mu\Delta t + O(\Delta t) \quad ...(10.53)$$

or
$$\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{O(\Delta t)}{\Delta t}.$$

Now, taking limit on both sides as $\Delta t \to 0$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \text{ for } n = 0 \quad ...(10.54)$$

Since only the steady state probabilities are considered here (see **sec. 10.4**),

$$\lim_{t\to\infty}\frac{d\,[P_n(t)]}{dt} = 0, \text{ for } n \ge 0 \text{ and } \lim_{t\to\infty} P_n(t) = P_n \text{ (which is independent of } t)$$

Consequently, the equations (10.52) and (10.54) can be written as :

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad \text{if } n > 0 \quad ...(10.52a)$$

$$0 = -\lambda P_0 + \mu P_1, \quad \text{if } n = 0 \quad ...(10.54a)$$

In this way, the equations (10.52a) and (10.54a) constitute the *system of steady state difference equations* for this model.

Q. In a single server, Poisson arrival and exponential service time queueing system, show that probability P_n of n customers in steady state satisfy the following equations :

$$\lambda P_0 = \mu P_1,\ (\lambda+\mu)P_1 = \mu P_2 + \lambda P_0, \text{ and } (\lambda+\mu)P_n = \mu P_{n+1} + \lambda P_{n-1}, \text{ for } n \ge 2.$$

II. To solve the system of difference equations. **[Meerut 97 P]**

By the technique of successive substitution. we solve the difference equations :

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad \text{if } n > 0,$$

$$0 = -\lambda P_0 + \mu P_1, \quad \text{if } n = 0,$$

Since $P_0 = P_0$

$P_1 = \frac{\lambda}{\mu}P_0$ [from the equation (10.54a) for $n = 0$]

$P_2 = \frac{\lambda}{\mu}P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$ [from letting $n = 1$ in the eqn. (10.52*a*) for $n > 0$ and substituting for P_1]

$P_3 = \frac{\lambda}{\mu}P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$ [from letting $n = 2$ in eqn. (10.52*a*) for $n > 0$ and substituting for P_2]

$\vdots \quad \vdots \quad \vdots$

$$P_n = \frac{\lambda}{\mu}P_{n-1} = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad [\text{for } n \ge 0] \quad ...(10.55)$$

Now using the fact that $\sum_{n=0}^{\infty} P_n = 1$,

$$P_0 + \frac{\lambda}{\mu}P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \dots + \left(\frac{\lambda}{\mu}\right)^n P_0 + \dots = 1 \quad \text{or} \quad P_0\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 \dots\right] = 1$$

or $$P_0\left[\frac{1}{1-\lambda/\mu}\right]=1$$ [since $(\lambda/\mu)<1$ as explained in **sec. 10.6**, sum of infinite G.P. is valid]

or $$P_0=1-(\lambda/\mu). \qquad ...(10.56)$$

Now, substituting the value of P_0 from (10.56) in (10.55), we get

$$P_n=\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)=\rho^n(1-\rho),\ \left(\rho=\frac{\lambda}{\mu}<1, n\geq 0\right) \qquad ...(10.57)$$

Thus, equations (10.56) and (10.57) rather give the required *probability distribution of queue length.*

Q. 1. Derive the differential-difference equations for the queueing model $(M \mid M \mid 1) : (\infty \mid FCFS)$. How would you proceed to solve the model ? **[Meerut (Stat.) 98; Delhi M.A/M.Sc. (OR). 90]**

2. Obtain the steady state solution of $(M \mid M \mid 1) : (\infty \mid \text{FCFS})$ system and also find expected value of queue length n. **[Meerut (Maths.) 97P; Garhwal (Stat.) 96]**

3. Explain $(M \mid M \mid 1) : (\infty \mid \text{FCFS})$ queueing model, derive and solve the difference equations in steady state, of the model. **[Agra 93; Garhwal M.Sc. (Stat.) 93]**

4. Show that for a single service station, Poisson arrivals and exponential service time, the probability that exactly n calling units are in the queueing system is $P_n=(1-\rho)\,\rho^n$, $n\geq 0$, where ρ is the traffic intensity.

Further, we may also compute a useful probability, *viz.*,

$$\text{Prob. }[\textit{queue size} \geq N]=\sum_{n=N}^{\infty}P_n=\sum_{n=0}^{\infty}P_n-\sum_{n=0}^{N-1}P_n=1-(P_0+P_1+\ldots+P_{N-1})$$

$$=1-\left[P_0+\frac{\lambda}{\mu}P_0+\ldots+\left(\frac{\lambda}{\mu}\right)^{N-1}P_0\right]=1-P_0\left[\frac{1-(\lambda/\mu)^N}{1-\lambda/\mu}\right]$$

$$=1-\left(1-\frac{\lambda}{\mu}\right)\left[\frac{1-(\lambda/\mu)^N}{1-\lambda/\mu}\right]=\left(\frac{\lambda}{\mu}\right)^N \qquad \text{[from (5.56), } P_0=1-(\lambda/\mu)]$$

$$\therefore \quad \textbf{Prob. } [\textit{queue size} \geq N]=(\lambda/\mu)^N=\rho^N. \qquad ...(10.58)$$

Q. 1. Describe a queue model and steady state equations of $M \mid M \mid 1$ queues. What is the prob. that at least one unit is present in the system. **[Meerut (I.P.M.) 90]**

2. Explain $M \mid M \mid 1$ queue model in the transient state. Derive steady state solution for the $M \mid M \mid 1$ queue model. **[Garhwal M.Sc. (Math.) 91]**

3. If P_n represents the probability of finding n in the long run in a queueing system with Poisson arrivals having parameter λ and exponential service times with parameter μ, show that

$$\lambda P_{n-1}-(\lambda+\mu)P_n+\mu P_{n+1}=0 \quad \text{for } n>0$$

and $$-\lambda P_0+\mu P_1=0. \quad \text{for } n=0$$

Solve these difference equations and obtain P_n in terms of $P=\lambda/\mu$. **[I.A.S. (Main) 95]**

III. To obtain probability density function of waiting time (excluding service time) distribution. **[Kanpur 93]**

In the steady state, each customer has the same waiting time distribution. This is a continuous distribution with probability density function $\Psi(w)$, and we denote by $\Psi(w)\,dw$ the probability that a customer begins to be served in the interval $(w, w+dw)$, where w is measured from the time of his arrival. We suppose that a customer arrives at time $w=0$ and service begins in the interval $(w, w+dw)$. *Fig.* 10.17 illustrates this situation. For convenience, we label the customer as A.

(*i*) There is a finite probability that waiting time is zero (P_0 the probability that the system is empty).

(*ii*) If there are n customers already in the system when the customer A *arrives*, n must leave before the service of A

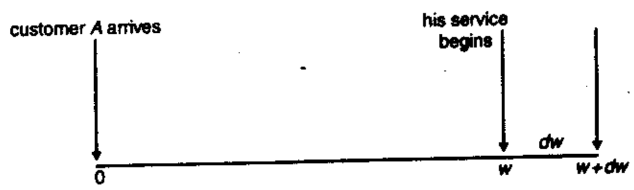

Fig. 10.17

begins. More precisely, $(n-1)$ customers must leave during the time interval $(0, w)$, and the nth customer during $(w, w+dw)$.

[If n customers left by the time w, service of A could begin before the interval (w, w + dw); and if fewer than (n − 1) left by time w, service could only begin in (w, w + dw); if there were two or more departures in that interval, the probability is O (dt) which may be ignored].

The server's mean rate of service is μ in unit time, or μw in time w, and the probability of $(n-1)$ departures in time w, during which the sever is busy, is the appropriate term of the Poisson distribution $(\mu w)^{n-1} e^{-\mu w}/(n-1)\,!$.

Let there be n units in the system (see *Fig.* 10.18), then

$\Psi_n(w)\,dw =$ Prob. $[(n-1)$ *units are served at time w]* × Prob. *[one unit is served in time dw]*,

or
$$\Psi_n(w)\,dw = \frac{(\mu w)^{n-1} e^{-\mu w}}{(n-1)\,!} \times \mu\, dw. \qquad \text{...(10.59)}$$

Let W be the waiting time of a unit who has to wait such that $w \le W \le w\,dw$, then the probability $\Psi(w)\,dw$ is given by

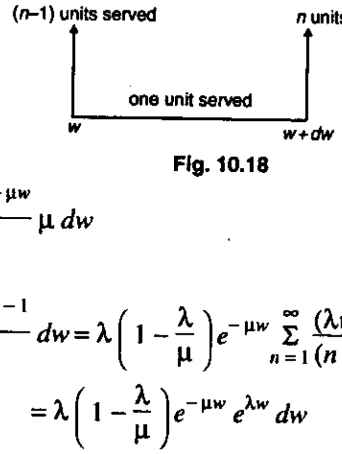

Fig. 10.18

$\Psi(w)\,dw =$ Prob. $(w \le W \le w + dw)$

= (The probability of n customers in the system when customer A arrives) × [the probability that exactly $n-1$ customers leave in $(0, w)$] × [the probability that nth customer leaves in $(w, w+dw)$], summed over all n from 1 to ∞

$$= \sum_{n=1}^{\infty} P_n \Psi_n(w)\,dw = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) \cdot \frac{(\mu w)^{n-1} e^{-\mu w}}{(n-1)\,!}\,\mu\, dw$$

[from (10.57) and (10.59)]

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)\mu\, e^{-\mu w} \sum_{n=1}^{\infty} \frac{[(\lambda/\mu)\,(\mu w)]^{n-1}}{(n-1)\,!}\,dw = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\mu w} \sum_{n=1}^{\infty} \frac{(\lambda w)^{n-1}}{(n-1)\,!}\,dw$$

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\mu w}\left[1 + \frac{(\lambda w)}{1\,!} + \frac{(\lambda w)^2}{2\,!} + \ldots\right]dw \qquad = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\mu w} e^{\lambda w}\,dw$$

$$\therefore \qquad \Psi(w) = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}, \; w > 0. \qquad \text{...(10.60)}$$

This result may also be obtained by using the *Laplace transform* of service time distribution, and the properties of the waiting time distribution may be found from its *Laplace transform.*

Obviously, $\int_0^{\infty} \Psi(w)\,dw \ne 1$, because it has the value λ/μ.

It is important to note that the case for which $w = 0$ has been excluded in eqn. (10.60). Thus,

Prob. $[W = 0]$ = Prob. *[no unit in the system]* $= P_0 = 1 - (\lambda/\mu)$. [from eqn. (10.56)]

Now, the sum of all probabilities of waiting time

$$= \int_0^{\infty} \text{Prob. } [w \le W \le w + dw] + \text{Prob. } [W = 0] \qquad \text{...(10.61)}$$

$$= \int_0^{\infty} \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,dw + \left(1-\frac{\lambda}{\mu}\right)$$

$$= \frac{\lambda}{\mu} + \left(1-\frac{\lambda}{\mu}\right) = 1$$

Hence it is concluded that the complete distribution for waiting time is partly continuous and partly discrete :

(*i*) ***continuous*** for $w \le W \le w + dw$ with probability density function $\Psi(w)$ given by eqn. (10.60); and

(*ii*) ***discrete*** for $W = 0$, with Prob $(W = 0) = 1 - (\lambda/\mu)$.

The probability that waiting time exceeds w is given by

$$\int_w^\infty \Psi(w)\,dw = \int_w^\infty \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,dw = \left(-\frac{\lambda}{\mu}e^{-(\mu-\lambda)w}\right)_w^\infty = \frac{\lambda}{\mu}e^{-(\mu-\lambda)w} = \rho e^{-(\mu-\lambda)w}$$

which does not include the service time.

Q. 1. Define cumulative probability distribution of waiting time for a customer who has to wait and show that in an $(M \mid M \mid 1) : (\infty \mid FIFO)$ queue system, it is given by $1-\rho\, e^{-\mu t(1-\rho)}$ where $\rho = \lambda/\mu$.

[Hint. Cum. Distribution $= \int_0^t \lambda(1-\rho)\, e^{-\mu(1-\rho)w}\,dw = (1-\rho)$.]

2. Define (M | M | 1) system. **[IGNOU 99 (Dec.)]**

IV. To find prob. distribution of time spent in the system (busy period distribution).

[Kanpur M.Sc. (Math.) 93]

In order to find the probability density function for the distribution of total time (waiting + service) an arrival spends in the system, let $\Psi(w \mid w > 0)$ = probability density function for waiting time such that *a person has to wait.*

The statement *"person has to wait"* is meant that the server remains busy in the *busy period.*

Applying the rule of conditional probability,

$$\Psi(w \mid w>0)\,dw = \frac{\Psi(w)\,dw}{\text{Prob. }(w>0)} = \frac{\Psi(w)\,dw}{\int_0^\infty \Psi(w)\,dw}.$$

Substituting the value for $\Psi(w)$ from eqn. (10.60),

$$\Psi(w \mid w>0)\,dw = \frac{\lambda(1-\lambda/\mu)\,e^{-(\mu-\lambda)w}\,dw}{\int_0^\infty \lambda(1-\lambda/\mu)\,e^{-(\mu-\lambda)w}\,dw} = \frac{\lambda(1-\lambda/\mu)\,e^{-(\mu-\lambda)w}\,dw}{\lambda/\mu}$$

or

$$\Psi(w \mid w>0) = (\mu-\lambda)\,e^{-(\mu-\lambda)w} \qquad ...(10.62)$$

Here, $\int_0^\infty \Psi(w \mid w>0)\,dw = \int_0^\infty (\mu-\lambda)\,e^{-(\mu-\lambda)w}\,dw = 1.$

Hence it gives the required probability density function for the busy period.

If service time is included then,

$$P(W \ge w) = \int_w^\infty (\mu-\lambda)\,e^{-(\mu-\lambda)w}\,dw = e^{-(\mu-\lambda)w}$$

V. MEASURES OF MODEL I :

(*i*) To find expected (average) number of units in the system, L_s.

By definition of expected value,

$$L_s = \sum_{n=1}^\infty n P_n = \sum_{n=1}^\infty n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right) = \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\sum_{n=1}^\infty n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\infty\right] = \left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[\frac{1}{(1-\lambda/\mu)^2}\right]^* \text{ (see foot-note)}$$

---

* Let $S = 1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\infty$, which is Arithmatico-Geometric series,

$\therefore \quad \frac{\lambda}{\mu}S = \left(\frac{\lambda}{\mu}\right)+2\left(\frac{\lambda}{\mu}\right)^2+\ldots\infty$

On subtracting,

$\left(1-\frac{\lambda}{\mu}\right)S = 1+\left(\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2+\ldots\infty = \frac{1}{1-\lambda/\mu}$ (sum of infinite G.P.)

$\therefore \quad S = \frac{1}{(1-\lambda/\mu)^2}.$

or $$L_s = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\rho}{1-\rho}, \text{ where } \rho = \lambda/\mu < 1 \quad ...(10.63)$$

which is the required formula.

Q. 1. In a certain queueing system with one server, the arrivals obey a Poisson distribution with mean λ and the service time distribution has mean $1/\mu$. Obtain the generating function of the length of the queue which a departing customer leaves behind him.

2. Show that for a single service station, Poisson arrivals and exponential service time, the probability that exactly *an calling units are in queueing system is* $P_n = (l-\rho)\rho^n$, $n \geq 0$ (ρ *is the traffic intensity*). Also, find the expected line length. **[B.H.U. 93]**

3. Show that average number of units in a $M \mid M \mid 1$ system is equal to $\rho/(1-\rho)$. **[Agra 99; Raj. Univ. (M. Phil) 93]**

4. Discuss (M I M I 1) : (∞ I FCFS) queueing model and find the expected line length $E(L_s)$ in the system. **[Garhwal M.Sc. (Math.) 96]**

5. For the M I M I 1 queueing system, find :
(a) Expected value of queue length n
(b) Prob. distribution of waiting time w. **[Meerut M.Sc. (Math.) BP-96]**

6. Explain a system with Poisson input, exponential waiting time with single chennel. Also determining the average length of the waiting time. **[Meerut 2002]**

(*ii*) To find expected (average) queue length, L_q : **[IGNOU 1999; Meerut 97 P, 93; Kanpur 93]**

Since there are $(n-1)$ units in the queue excluding one being serviced,

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right] = L_s - [1-P_0] \quad \left(\text{since } \sum_{n=0}^{\infty} P_n = 1\right)$$

Substituting the value of P_0 from (10.56), we have

$$L_q = L_s - 1 + \left(1 - \frac{\lambda}{\mu}\right) \quad ...(10.64)$$

or $$L_q = L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}, \text{ where } L_s = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\rho}{1-\rho}.$$

Q. 1. For $M \mid M \mid 1$ model, obtain the expected number of waiting customers if the queueing process is going on for a long time.

2. Describe the system of steady state equations for a queueing model ($M \mid M \mid 1$) : (*FCFS*, ∞) and obtain their solution. Obtain the mean queue length and mean number of units in the system. **[Meerut 95, 90]**

(*iii*) To find mean (or expected) waiting time in the queue (excluding service time), W_q :

Since expected time an arrival spends in the queue is given by

$$W_q = \int_0^{\infty} w\Psi(w)\,dw = \int_0^{\infty} w.\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,dw \quad \text{[from eqn. (10.60)]}$$

Integrating by parts,

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\left[w\cdot\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)} - \frac{1}{(\mu-\lambda)^2}e^{-(\mu-\lambda)w}\right]_0^{\infty} = \lambda\left(\frac{\mu-\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2}$$

$$\therefore \quad W_q = \frac{\lambda}{\mu(\mu-\lambda)} \quad ...(10.65)$$

(*iv*) To find expected waiting time in the system (including service time), W_s :

Since expected waiting time in the system = Expected waiting time in queue + expected service time, i.e. $W_s = W_q + 1/\mu$ (expected service time or mean service time = $1/\mu$). Substituting the value of W_q from eqn. (10.65), we get

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda} \quad ...(10.66)$$

(*v*) To find expected waiting time of a customer who has to wait, $(W \mid W > 0)$.

The expected length of the busy period is given by

$$(W \mid W > 0) = \int_0^\infty w \Psi(w > 0)\, dw = \int_0^\infty w \,.\, (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw \quad \text{[from eqn. (10.62)]}$$

Integrating by parts, we get

$$(W \mid W > 0) = \frac{1}{\mu - \lambda} = \frac{1}{\mu\,(1 - \rho)} \qquad \text{...(10.67)}$$

(*vi*) **To find expected length of non-empty queue, $(L \mid L > 0)$.**

By definition of conditional probability,

$(L \mid L > 0) = L_s /$ Prob. (an arrival has to wait, $L > 0$)

$= L_s/(1 - P_0)$ (since probability of an arrival not to wait is P_0)

Substituting the value of L_s and P_0 from eqns. (5.63) and (10.56), we get

$$(L \mid L > 0) = \frac{(\lambda/\mu)/(1 - \lambda/\mu)}{\lambda/\mu} = \frac{\mu}{\mu - \lambda} = \frac{1}{1 - \rho} \qquad \text{...(10.68)}$$

(*vii*) **To find the variance of queue length.** **[Kanpur 93]**

By definition,

$$\text{Var.}\{n\} = \sum_{n=1}^{\infty} n^2 P_n - \left(\sum_{n=1}^{\infty} n P_n \right)^2 = \sum_{n=1}^{\infty} n^2 P_n - [L_s]^2$$

$$= \sum_{n=1}^{\infty} n^2 (1 - \rho)\, \rho^n - \left(\frac{\rho}{1 - \rho} \right)^2 \qquad \text{(using (10.57) and (10.63))}$$

$$= (1 - \rho)\, [1^2 \rho + 2^2 \rho^2 + 3^2 \rho^3 + \ldots] - \rho^2/(1 - \rho)^2$$

$$= (1 - \rho)\, \rho\, [1 + 2^2 \rho + 3^2 \rho^2 + \ldots] - \rho^2/(1 - \rho)^2$$

$$= (1 - \rho)\, \rho \left[\frac{1 + \rho}{(1 - \rho)^3} \right]^* - \frac{\rho^2}{(1 - \rho)^2} \qquad \textbf{[see foot note]}$$

$$= \rho/(1 - \rho)^2.$$

Q. 1. Obtain the steady state equations for the model $(M \mid M \mid 1) : (\infty \mid FCFS)$ *i.e.* single server, Poisson arrival, negative exponential service), and also find the formula for :

(*i*) variance of the queue length, (*ii*) the average waiting length, (*iii*) Prob. Queue size $\geq N$,

(*iv*) The average (mean) queue length., (*v*) the average waiting length given that it is greater than zero,

(*vi*) The average number of customers in the system. **[Raj. Univ. (M. Phil) 90]**

2. Define the concept of busy period in queueing theory and obtain its distribution for the system $M \mid M \mid 1 : (\infty \mid FCFS)$.

Show that the average length of the busy period is $1/(\mu - \lambda)$.

3. Customers arrive at a sales counter in a Poisson fashion with mean arrival rate λ and exponential service times with mean service rate μ. Determine :

(*i*) Average length of non-empty queues, (*ii*) Average waiting time of an arrival.

4. For the queueing system in which there is a single channel and the inter-arrival time of units and the service time of units follow exponential distribution prove that :

(*i*) $1/(\mu - \lambda)$ is the average time an arrival spends in the system, (*ii*) $\lambda^2/\mu\,(\mu - \lambda)$ is average queue length.

(*viii*) **To find the probability of arrivals during the service time of any given customer.**

Since the arrivals are Poisson and service times are exponential, the probability of r arrivals during the service time of any given customer is given by

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* Let $S = 1 + 2^2\rho + 3^2\rho^2 + \ldots$

Integrating both sides in the limit 0 to ρ

$$\int_0^\rho S\, d\rho = \rho + 2\rho^2 + \ldots = \rho(1 - \rho)^{-2}$$ [see foot-note on p. 451]

Now, differentiating w.r.t. 'ρ'

$$S = \frac{1}{(1 - \rho)^2} + \frac{2\rho}{(1 - \rho)^3} = \frac{(1 + \rho)}{(1 - \rho)^3}.$$

$$K_r = \int_0^\infty P_r(t)\, s(t)\, dt = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{r!} \cdot \mu e^{-\mu t}\, dt = \frac{\lambda^r \mu}{r!} \int_0^\infty e^{-(\lambda+\mu)t}\, t^r\, dt$$

$$= \frac{\lambda^r \mu\, \Gamma(r+1)}{r!\,(\lambda+\mu)^{r+1}} \quad \left[\text{using } \int_0^\infty e^{-at}\, t^n\, dt = \frac{\Gamma(n+1)}{a^{n+1}}\right]$$

$$= \left(\frac{\lambda}{\lambda+\mu}\right)^r \cdot \frac{\mu}{\lambda+\mu} \qquad [\text{since } \Gamma(r+1) = r!]$$

10.12-1. Inter-Relationship Between L_s, L_q, W_s, W_q.

It can be proved under (rather) general conditions of arrival, departure, and service discipline that the formulae,

$$L_s = \lambda W_s, \qquad ...(10.69)$$

and

$$L_q = \lambda W_q, \qquad ...(10.70)$$

will hold in general. These formulae act as key points in establishing the strong relationships between W_s, W_q, L_s and L_q which can be found as follows.

By definition, $W_q = W_s - 1/\mu$. ...(10.71)

Thus, multiplying both sides by λ and substituting the values from (10.69) and (10.70),

$$L_q = L_s - \lambda/\mu. \qquad ...(10.72)$$

This means that one of the four expected values (together with λ and μ) should immediately yield the remaining three values.

Q. In (M | M | 1) : (∞ | FCFS) model obtain p.d.f. of waiting time (excluding service time) and hence obtain $E(W_q)$, $E(W_s)$, $E(L_q)$, $E(L_s)$. **[Garhwal M.Sc. (Stat.) 93]**

10.12-2. Illustrative Examples on Model I

Example 1. *A TV repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?*

[JNTU 2002; Agra 98, 93; Karnataka B.E. (CSE) 93; Meerut (Maths.) 91]

Solution. Here, $\mu = 1/30$, $\lambda = 10/(8 \times 60) = 1/48$. Therefore, expected number of jobs are

$$L_s = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} = \frac{1/48}{1/30 - 1/48} = 1\tfrac{2}{3} \text{ jobs.} \qquad \textbf{Ans.}$$

Since the fraction of the time the repairman is busy (*i.e.* traffic intensity) is equal to λ/μ, the number of hours for which the repairman remains busy in a 8-hour day is

$$= 8\,.\,(\lambda/\mu) = 8 \times 30/48 = 5 \text{ hours.}$$

Therefore, the time for which the repairman remains idle in 8-hour day = (8 – 5) hours = 3 hours. **Ans.**

Example 2. *At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes ? It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.* **[Meerut (Maths.) 99, 96]**

Solution. Here, $\lambda = 15/60 = 1/4$ customer/minute, $\mu = ?$ Prob. [waiting time ≥ 12] = 1 – 0.90 = 0.10.

Therefore, $$\int_{12}^\infty \lambda\left(1-\frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)w}\, dw = 0.10 \quad \text{or} \quad \lambda\left(1-\frac{\lambda}{\mu}\right)\left[\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)}\right]_{12}^\infty = 0.10$$

or $$e^{(3-12\mu)} = 0.4\,\mu \quad \text{or} \quad 1/\mu = 2.48 \text{ minute per service.} \qquad \textbf{Ans.}$$

Example 3. *Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call assumed to be distributed exponentially with mean 3 minutes. Then,*

(*a*) *What is the probability that a person arriving at the booth will have to wait ?*
[Meerut 2002; I.A.S. (Main) 91]

(*b*) *What is the average length of the queues that form from time to time ?* **[Meerut 2002; Kanpur 2000]**

(*c*) *The telephone department will install a second booth when convinced that an arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth ?* **[Meerut 2002; I.A.S. (Maths) 91]**

Solution. Here, $\lambda = 1/10$ and $\mu = 1/3$

(*a*) Prob. $(W > 0) = 1 - P_0 = \lambda/\mu$ [from eqn. (10.60)] $= \frac{1}{10} \times \frac{3}{1} = \frac{3}{10} = 0.3$ **Ans.**

(*b*) $(L \mid L > 0) = \mu/(\mu - \lambda) = \frac{1}{3}/(\frac{1}{3} - \frac{1}{10}) = 1.43$ persons [from eqn. (10.68)] **Ans.**

(*c*) $W_q = \lambda/\mu(\mu - \lambda)$ [from eqn. (10.65)]

Since $W_q = 3$, $\mu = \frac{1}{3}$, $\lambda = \lambda'$ (say) for second booth, therefore

$$3 = \frac{\lambda'}{\frac{1}{3}(\frac{1}{3} - \lambda')}, \quad \text{giving } \lambda' = 0.16.$$

Hence, increase in the arrival rate $= 0.16 - 0.10 = 0.06$ arrivals per minute. **Ans.**

Example 4. *As in Example 3, a telephone booth with Poisson arrivals spaced 10 minutes apart on the average, and exponential call lengths averaging 3 minutes.*

(*a*) *What is the probability that an arrival will have to wait more than 10 minutes before the phone is free ?*

(*b*) *What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call ?*

(*c*) *Estimate the fraction of a day that the phone will be in use.*

(*d*) *Find the average number of units in the system.*

Solution. Here $\lambda = 0.1$ arrival per minute, $\mu = 0.33$ service per minute.

(*a*) Prob. [waiting time ≥ 10] $= \int_{10}^{\infty} \Psi(w)\, dw = \int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu - \lambda)w}\, dw$

$$= -\frac{\lambda}{\mu}\left[e^{-(\mu - \lambda)w}\right]_{10}^{\infty} = 0.3e^{-2.3} = 0.03 \text{ [\textit{see Exponential Tables}]}$$ **Ans.**

(*b*) Prob. [waiting time in the system ≥ 10]

$$= \int_{10}^{\infty} (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw = e^{-10(\mu - \lambda)} = e^{-2.3} = 0.10 \text{ [\textit{see Exp. Table}]}$$ **Ans.**

(*c*) The fraction of a day that the phone will be busy = traffic intensity $\rho = \lambda/\mu = 0.3$.

(*d*) Average number of units in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{10}}{\frac{1}{3} - \frac{1}{10}} = \frac{3}{7} = 0.43 \text{ customer.}$$ **Ans.**

Example 5. (a) *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average 36 minutes. Calculate the following :*

(*i*) *The average number of trains in the queue.*

(*ii*) *The probability that the queue size exceeds 10.* **[JNTU 99, 98; Raj. Univ. (M. Phil.) 93, 90]**

If the input of trains increases to an average 33 per day, what will be change in (i) and (ii) ?

Establish the formula you use in your calculations. **[JNTU 2002; Agra 98; I.A.S. (Main) 90]**

Solution. Here

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute; } \mu = \frac{1}{36} \text{ trains/minute, and } \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75.$$

(*i*) $$L_s = \frac{\rho}{1 - \rho} = \frac{0.75}{1 - 0.75} = 3 \text{ trains}.$$

(*ii*) Prob [*queue size* ≥ 10] $= \rho^{10} = (0.75)^{10} = 0.06$. **Ans.**

When the input increases to 33 trains per day $\lambda = \frac{1}{43}$, $\mu = \frac{1}{36}$.

Therefore, $\rho = \lambda/\mu = 36/43 = 0.84$. Hence,

(*i*) $L_s = 0.84/0.16 = 5$ trains (*ii*) Prob (queue size ≥ 10) $= (0.84)^{10} = 0.2$ (approx.)

(b) *Trains arrive at the yard every 20 minutes and the service time is 40 minutes. If the line capacity of the yard is limited to 6, find*

(i) the probability the yard is empty.

(ii) the average number of trains in the system. **[JNTU (B. Tech.) 2003]**

Solution. Proceed as in part (a).

Example 6. *In the above problem calculate the following :*

(*i*) *Expected waiting time in the queue.*

(*ii*) *The probability that number of trains in the system exceeds 10.* **[JNTU (B. Tech.) 98]**

(*iii*) Average number of trains in the queue.

Solution. Here $\lambda = 1/48$, $\mu = 1/36$ and $\rho = 0.75$.

(*i*) Expected waiting time in the queue is

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/48}{1/36\,(1/36 - 1/48)} = 108 \text{ minutes } \textit{or 1 hr 48 mts.}$$

(*ii*) $P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06$.

(*iii*) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{(1/48)^2}{1/36\,(1/36 - 1/48)} = \dfrac{108}{48} = 2.25$ or nearly 2 trains.

Example 7. *Consider an example from a maintenance shop. The inter-arrival times at toolcrib are exponential with an average time of 10 minutes. The length of the service time (amount of time taken by the toolcrib operator to meet the needs of the maintenance man) is assumed to be exponentially distributed, with mean 6 minutes. Find :*

(*i*) *The probability that a person arriving at the booth will have to wait.*

(*ii*) *Average length for the queue that forms and the average time that an operator spends in the Q-system.*

(*iii*) *The manager of the shop will install a second booth when an arrival would have to wait 10 minutes or more for the service. By how much must the rate of arrival be increased in order to justify a second booth.*

(*iv*) *The probability that an arrival will have to wait for more than 12 minutes for service and to obtain his tools.*

(*v*) *Estimate the fraction of the day that toolcrib operator will be idle.*

(*vi*) *The probability that there will be six or more operators waiting for the service.*

Solution. Here $\lambda = 60/10 = 6$ per hour, $\mu = 60/6 = 10$ per hour **[Virbhadrah 2000]**

(*i*) *A person will have to wait if the service facility is not idle.*

Probability that the service facility is idle = Probability of no customer in the system (P_0)

$\therefore$ Probability of waiting $= 1 - P_0 = 1 - (1 - \rho) = \rho = \lambda/\mu = 6/10 = 0.6$ **Ans.**

(*ii*) $L_q = \rho^2/(1 - \rho) = (0.6)^2/(1 - 0.6) = 0.9$

$L_s = L_q + \lambda/\mu = 0.9 + 0.6 = 1.5$. $\therefore$ $W_s = L_s/\lambda = 1.5/6 = 1/4$ hours

(*iii*) $W_q = L_q/\mu = 0.9/6$ hrs. $= 9$ minutes.

Let λ be the arrival rate when a second booth is justified, *i.e.*, $W_q \geq 10$ minutes.

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{60}.$$

$\therefore$ $6\lambda = 10(10 - \lambda)$ or $16\lambda = 100$ or $\lambda = 6.25$.

Hence if the arrival rate exceeds 6.25 per hour, the second booth will be justified.

(*iv*) Probability of waiting for 12 minutes or more is given by

$$\text{Prob. } (W \geq 12) = \int_{12}^{\infty} \rho(\mu - \lambda)\, e^{-(\mu - \lambda)w} dw = -\rho\left[e^{-(\mu - \lambda)w}\right]_{12/60}^{\infty}$$

$$= \rho e^{-(\mu - \lambda)12/60} = 0.6\, e^{-(10 - 6).12/60} = 0.6\, e^{-4/5} = 0.27.$$ **Ans.**

(*v*) $P_0 = 1 - \rho = 0.4$, 40% of the time of toolcrib operator is idle.

(*vi*) Probability of six or more operators waiting for the service $= \rho^6 = (0.6)^6$.

Example 8. *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from 1⅓ patients to ½ patient.* **[JNTU (B. Tech.) 2004; I.A.S. (Main) 93]**

Solution. Here $\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$ patient/minute, $\mu = \frac{1}{10}$ patient/minute

Expected number of patients in the waiting line

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(1/15)^2}{1/10\,(1/10 - 1/15)} = 1\tfrac{1}{3} \text{ patients.}$$

But, $L_q = 1\frac{1}{3}$ is reduced to $L_q' = \frac{1}{2}$.

Therefore, substituting $L_q' = 1/2$, $\lambda' = \lambda = 1/15$ in the formula $L_q' = \frac{\lambda'^2}{\mu'(\mu' - \lambda')}$, we get

$$1/2 = \frac{(1/15)^2}{\mu'(\mu' - 1/15)}$$

which gives $\mu' = 2/15$ patient/minute.

Hence the average rate of treatment required is $1/\mu' = 7.5$ minutes.

Consequently, the decrease in the average rate of treatment $= 10 - 15/2 = 5/2$ minutes;
and the budget per patient $= 100 + 5/2 \times 10 =$ Rs. 125. So in order to get the required size of the queue, the budget should be increased from Rs. 100 to Rs. 125 per patient.

Example 9. *The mean rate of arrival of planes at an airport during the peak period is 20 hour, but the actual number of arrivals in any hour follows a Poisson distribution with the respective averages. When there is congestion, the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.*

(*i*) *How many planes would be flying over the field in the stack on an average in good weather and in bad weather ?*

(*ii*) *How long a plane would be in the stack and in the process of landing in good and in bad weather ?*

(*iii*) *How much stack and landing time to allow so that priority to land out of order would have to be requested only one time in twenty ?* **[Agra 98]**

Solution. Here

$$\lambda = 20 \text{ planes/hour}$$

and

$$\mu = \begin{cases} 60 \text{ planes/hour in good weather} \\ 30 \text{ planes/hour in bad weather} \end{cases}$$

(*i*) $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \begin{cases} 20^2/60(60 - 20) = 1/6 \text{ in good weather} \\ 20^2/30(30 - 20) = 1/3 \text{ in bad weather} \end{cases}$ **Ans.**

(*ii*) $W_s = \frac{1}{\mu - \lambda} = \begin{cases} 1/(60 - 20) = 1/40 \text{ hrs. in good weather} \\ 1/(30 - 20) = 1/10 \text{ hrs. in bad weather} \end{cases}$ **Ans.**

(*iii*) The waiting time 'w' taken by the plane in landing and in the stack is given by

$$\int_0^w (\mu - \lambda)\, e^{-(\mu - \lambda)w} dw = 0.95$$

or

$$(\mu - \lambda)\left[\frac{e^{-(\mu - \lambda)w}}{\lambda - \mu}\right]_0^w = 0.95$$

or $1 - e^{-(\mu - \lambda)w} = 0.95$ or $e^{-(\mu - \lambda)w} = 0.05$ (A)

Now the value of w can be determined in both the cases.

Case I. *When weather is good* :

Substituting $\lambda = 20$, $\mu = 60$ in eqn. (A), $e^{-40w} = 0.05$.

Taking the logarithm of both sides to the base e (instead of 10),

$$-40w = \log_e (0.05) = -2.9957 \qquad [\because \log_e e = 1, \therefore \log_e (0.05) = -2.9957]$$

or
$$w = \frac{2.9957}{40} = 0.075 \text{ hour} = 4.5 \text{ minutes } \textbf{Ans.}$$

Case 2. *When weather is bad :*

Substitution $\lambda = 20, \mu = 30$ in eqn. (A),

$$e^{-10w} = 0.05$$

Solving this equation as in case I above, we get

$$w = 0.3 \text{ hour} = 18 \text{ minutes.}$$ **Ans.**

Example 10. *A refinery distributes its products by trucks, loaded at the loading dock. Both company-trucks and independent distributor's trucks are loaded. The independent firms complained that sometimes they must wait in line and thus lose money paying for a truck and driver, that is only waiting. They have asked the refinery either to put in a second loading dock or to discount prices equivalent to the waiting time. Extra loading dock cost Rs. 100/- per day whereas the waiting time for the independent firms cost Rs. 25/- per hour. The following data have been accumulated. Average arrival rate of all trucks is 2 per hour and average service rate is 3 per hour. Thirty per cent of all trucks are independent. Assuming that these rates are random according to the Poisson distrbutions, determine :*

(*a*) *the probability that a truck has to wait*

(*b*) *the waiting time of a truck that waits, and*

(*c*) *the expected cost of waiting time of independent trucks per day.*

Is it advantageous to decide in favour of a second loading dock to ward off the complaints ?

Solution. We are given that $\lambda = 2$ per hour and $\mu = 3$ per hour.

(*a*) The probability that a truck has to wait for service is the utilization factor,

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} = 0.66.$$

(*b*) The waiting time of a truck that waits is

$$(W \mid W > 0) = \frac{W_s}{\text{Prob }(W > 0)} = \left[\frac{\lambda}{\mu(\mu - \lambda)} / (\lambda/\mu)\right]$$
$$= \frac{1}{\mu - \lambda} = \frac{1}{3-2} = 1 \text{ hour.}$$

(*c*) The total expected waiting time of independent trucks per day is given by :

Expected waiting time = Trucks per day × % Independent truck × Expected waiting time per truck

$$= (2 \times 8)(0.3W_q) = 16 \times 0.3 \times \frac{\lambda}{\mu(\mu - \lambda)} = 4.8 \times \frac{2}{3(3-2)} = 3.2 \text{ hour per day.}$$

$\therefore$ Expected Cost = Rs. (3.2×25) = Rs. 80.

Example 11. (a) *Barber A takes 15 minutes to complete one hair cut. Customers arrive in his shop at an average rate of one every 30 minutes and the arrival process is Poisson. Barber B takes 25 minutes to complete one hair-cut and customers arrive in his shop at an average rate of one every 50 minutes, the arrival process being Poisson during steady state.*

(*i*) *Where would you expect the bigger queue ?*

(*ii*) *Where would you require more times waiting included, to complete a hair-cut.*

(b) *In a hair dressing salon with one barber the customer arrival follows Poission distribution at an average rate of one every 45 minutes. The service time is exponentially distributed with a mean of 30 minutes. Find (i) Average number of customers in the salon.*

(ii) Average waiting time of a customer before service.

(iii) Average idle time of the barber. **[VTU (BE Mech.) 2003]**

Solution. Proceed as in above solved examples.

Example 12. (a) *An airlines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of customers arriving during any given period is Poisson distributed with an arrival rate of eight per hour and that the reservation clerk can serve a customer in six minutes on an average, with an exponentially distributed service time.*

(i) What is the probability that the system is busy ?

(ii) What is the average time a customer spends in the system ?

(iii) What is the average length of the queue and what is the number of customers in the system ?

[C.A., Nov. 96]

(b) *If for a period of 2 hours in a day (8–10 AM) planes arrive at the aerodrome for every 20 minutes but the service time continues to remain 32 minutes then calculate for this period :*

(i) the probability that the aerodrome is empty (ii) average queue length on the assumption that the time capacity of the aerodrome is limited to 6 planes. **[JNTU (Mech. & Prod.) May 2004]**

Solution. (a) According to the given information :

Mean arrival rate, $\lambda = 8$ customers per hour

Mean service rate, $\mu = \frac{60}{6} = 10$ customers per hour

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{8}{10} \quad \text{or} \quad \frac{4}{5}$$

(i) The probability that the system is busy is given by :

$$1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = 0{\cdot}8$$, *i.e.*, 80% of the time system is busy.

(ii) The average time a customer spends in the system is given by :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{10 - 8} = \frac{1}{2} \text{ hours or 30 minutes}$$

(iii) The average length of the queue is given by :

$$L_q = \frac{\lambda}{\mu} \times \frac{\lambda}{\mu - \lambda} = \frac{8}{10} \times \frac{8}{10 - 8} = 3{\cdot}2 \text{ customers}$$

The average number of customers in the system is given by :

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{10 - 8} \text{ or 4 customers.}$$

(b) [Hint : Proceed as (a)**]**

Example 13. *Customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.* **[Poona (M.B.A.) 98]**

Solution. Here we are given :

$$\lambda = 20 \text{ per hour}, \mu = \frac{60 \times 60}{100} = 36 \text{ per hour}$$

The average waiting time of a customer in the queue is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{36(36 - 20)} = \frac{5}{36 \times 4} \text{ hours or } \frac{5 \times 3600}{36 \times 4}, \text{ i.e., 125 seconds.}$$

The average waiting time of a customer in the system is given by :

$$W_s = \frac{1}{(\mu - \lambda)} = \frac{1}{(36 - 20)} \text{ or } \frac{1}{16} \text{ hour i.e., 225 seconds.}$$

Example 14. *Customers arrive at a one-window drive according to a Poisson distribution with mean of 10 minutes and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have to wait outside this space. Calculate :*

(i) Probability that an arriving customer can drive directly to the space in front of the window.

(ii) Probability that an arriving customer will have to wait outside the directed space.

[JNTU (B. Tech.) 2003; SJMIT (BE Mech.) 2002; C.A. (May) 98]

(iii) How long an arriving customer is expected to wait before getting the service ?

Solution. From the given information, we find that :

Mean arrival rate, $\lambda = 6$ customers per hour

and mean service rate, $\mu = 10$ customers per hour

(i) Probability that an arriving customer can drive directly to the space in front of the window is given by :

$$P_0 + P_1 + P_2 = \left(1 - \frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right]$$

$$= \left(1 - \frac{6}{10}\right)\left[1 + \frac{6}{10} + \left(\frac{6}{10}\right)^2\right] = \frac{98}{1225} \text{ or } 0{\cdot}784$$

(ii) Probability that an arriving customer will have to wait outside the directed space is given by :

$$1 - (P_0 + P_1 + P_2) = 1 - 0{\cdot}784 = 0{\cdot}216 \text{ or } 21{\cdot}6\%$$

(iii) Expected waiting time of a customer being getting the service is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{6}{10(10-6)} = \frac{3}{20} \text{ hr. or 9 minutes.}$$

Example 15. *The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.*

(i) What is the probability that a person arriving at the booth will have to wait ?

(ii) What is the average length of the non-empty queues that form from time to time ?

(iii) The Mahanagar Telephone Nigam Ltd. will install a second booth when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth ?

(iv) Estimate the fraction of a day that the phone will be in use. **[C.A., (May) 1999; Delhi (M. Com.) 99]**

Solution. Here we are given :

$$\lambda = \frac{1}{10} \times 60 \text{ or 6 per hour and } \mu = \frac{1}{3} \times 60 \text{ or 20 per hour}$$

(i) Probability that a person arriving at the booth will have to wait

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0{\cdot}3.$$

(ii) Average length of non-empty queues

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 6} = 1{\cdot}42.$$

(iii) The installation of a second booth will be justified if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time :

$$W_q' = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{3}{60} = \frac{\lambda'}{20(20 - \lambda')} \text{ or } \lambda' = 10.$$

(ii) P_0 = Prob. of no customer in the system $= 1 - \dfrac{\lambda}{\mu} = 0{\cdot}5$

Thus 50% of time an arrival will not have to wait.

(iii) Average time spent by a customer $= \dfrac{1}{\mu - \lambda} = \dfrac{1}{5}$ hour or 12 minutes

(iv) Average queue length $= \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{5 \times 5}{10(10-5)} = 0{\cdot}5$

(v) The management will deploy the person exclusively for Xeroxing when the average time spent by a customer exceeds 15 minutes. We wish to calculate the arrival rate that will lead to such situation. Let this arrival rate be λ'. Then

$$\frac{1}{\mu - \lambda'} > \frac{15}{60} \text{ or } \frac{1}{10 - \lambda'} > \frac{1}{4}, \textit{i.e.}, \lambda' > 6.$$

Hence, if the arrival rate of customers is greater than 6 customers per hour, the average time spent by a customer will exceed 15 minutes.

Example 16. *Telephone users arrive at a booth following a Poisson distribution with an average time of 5 minutes between one arrival and the next. The time taken for a telephone call is on an average 3 minutes and it*

follows an exponential distribution. What is the probability that the booth is busy ? How many more booths should be established to reduce the waiting time to less than or equal to half of the present waiting time ?
[JNTU (B. Tech.) 2003; Nagpur (M.B.A.) Nov. 98; Madras (M.B.A.), Dec. 97]

Solution. Here we are given :

arrival rate, $\lambda = 12$ per hour

service rate, $\mu = 20$ per hour

Probability that booth is busy

$$= 1 - P_0 = \frac{\lambda}{\mu} = \frac{12}{20} = 0{\cdot}60$$

Average waiting time in queue :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{12}{20(20-12)} = \frac{3}{40}\text{ hour}$$

Average waiting time in system :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{20-12} = \frac{1}{8}\text{ hour}$$

In case waiting time is required to be reduced to half,

$$W_s' = \frac{1}{\mu' - \lambda} \Rightarrow \frac{1}{16} = \frac{1}{\mu' - 12} \text{ or } \mu' = 28 \text{ per hour.}$$

Example 17. *The XYZ company's quality control deptt. is managed by a single clerk, who takes on an average 5 minutes in checking parts of each of the machine coming for inspection. The machines arrive once in every 8 minutes on the average. One hour of the machine is valued at Rs. 15 and a clerk's time is valued at Rs. 4 per hour. What are the average hourly queuing system costs associated with the quality control department ?*
[Poona (M.B.A.) 95]

Solution. Mean arrival rate, $\lambda = \frac{1}{4}$ per min. $= \frac{60}{8}$ per hour

Mean service rate, $\mu = \frac{1}{5}$ per min. $= 12$ per hour

Average time spent by a machine in the system

$$= \frac{1}{\mu - \lambda} = \frac{2}{9}\text{ hour}$$

Average queuing cost per machine is $\frac{15 \times 2}{9}$ = Rs. $\frac{10}{3}$

Average arrival of $\frac{60}{8}$ machines per hour costs $\frac{10}{3} \times \frac{60}{8}$ or Rs. 25

Average hourly queuing cost = Rs. 25.

Average hourly cost for the clerk = Rs. 4

Hence total cost = Rs. 29 per hour.

Example 18. *A company distributes its products by trucks loaded at its only loading station. Both, company's trucks and contractor's trucks, are used for this purpose. It was found out that on an average every five minutes, one truck arrived and the average loading time was three minutes. 50% of the trucks belong to the contractor. Find out :*

(i) the probability that a truck has to wait,

(ii) the waiting time of truck that waits, and

(iii) the expected waiting time of contractor's trucks per day, assuming a 24-hours shift.
[Punjabi (M.B.A.) 99; C.A. Nov. 96]

Solution. Here we are given :

Average arrival rate of trucks, $\lambda = \frac{60}{5} = 12$ trucks/hr.

Average service rate of trucks, $\mu = \frac{60}{3} = 20$ trucks/hr.

(i) The probability that a truck has to wait is given by :

$$\rho = \frac{\lambda}{\mu} = \frac{12}{20} = 0{\cdot}6$$

(ii) The waiting time of a truck that waits is given by :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{20 - 12} = \frac{1}{8} \text{ hour or } 7{\cdot}5 \text{ minutes.}$$

(iii) The expected waiting time of contractor's truck per day (assuming 24 hrs. shift)

= (No. of trucks per day) × (Contractor's percentage) × (Expected waiting time of a truck)

$$= 12 \times 24 \times \frac{50}{100} \times \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= 288 \times \frac{1}{2} \times \frac{12}{20 \times 8} = \frac{54}{5} \text{ or } 10{\cdot}8 \text{ hrs.}$$

Example 19. *A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost of a truck is Rs. 20 per hour and the members of the loading crew are paid @ Rs. 6 each per hour. Would you advise the truck owner to add another crew of three persons ?* **[Delhi (M. Com.) 96]**

Solution. Total hourly cost = Loading crew cost + cost of waiting time

With present crew :

Loading cost = No. of loaders × Hourly wage rate = Rs. 18/hour.

Waiting time cost

$$= \left[\begin{array}{c}\textit{Expected waiting}\\ \textit{time per truck } (W_s)\end{array}\right] \times \left[\begin{array}{c}\textit{Expected arrivals}\\ \textit{per hour } (\lambda)\end{array}\right] \times \left[\begin{array}{c}\textit{Hourly}\\ \textit{waiting cost}\end{array}\right]$$

$$= \frac{4}{6-4} \times 20 = \text{Rs. } 40/\text{hr.}$$

Total cost = Rs. 18 + Rs. 40 = Rs. 58/hr.

After proposed crew addition :

$$\text{Total cost} = 6 \times 6 + \frac{4}{(12-4)} \times 20 = \text{Rs. } 46/\text{hr.}$$

Hence it will be beneficial to add a crew of 3 loaders.

Example 20. *A road transport company has one reservation clerk on duty at a time. She handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following :*

(i) What is the average number of customers waiting for the service of the clerk ?

(ii) What is the average time a customer has to wait before getting service ?

(iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to be 12 paise per minute spent waiting before being served, should the company install the computer system ? Assume 8-hour working day. **[Madras (M.B.A.) 97]**

Solution. (i) Here we are given :

customer arrival rate = λ = 8 per hour and

service rate = μ = 12 per hour.

Average number of customers waiting for the service of the clerk (in the system) :

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{12 - 8} = 2 \text{ customers.}$$

The average number of customers waiting for the service of the clerk (in the queue) :

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{8 \times 8}{12(12-8)} \text{ or } 1{\cdot}33 \text{ customers.}$$

(ii) The average waiting time of a customer (in the system) before getting service :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12 - 8} \text{ hour or } 15 \text{ minutes.}$$

(iii) The average waiting time of a customer (in the queue) before getting service :

$$W_s = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{ hour or 10 minutes.}$$

(iv) Here calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

Now an arrival waits for W_q hours before being served and there are λ arrivals per hour. Thus expected waiting time for all customers in an 8 hour day with one system $= 8\lambda.\ W_q = 8 \times 8 \times \frac{1}{6}$ hrs. or $\frac{64}{6} \times 60$ minutes, *i.e.*, 640 minutes.

The goodwill cost per day with one system

$$= \text{Rs. } 640 \times \frac{12}{100} = \text{Rs. } 76{\cdot}80$$

The expected waiting time of a customer before getting service when there is an additional computer system is :

$$W_q' = \frac{8}{20(20-8)} = \frac{8}{20 \times 12} \text{ or } \frac{1}{30} \text{ hr.}$$

Thus expected waiting time of customers in an 8-hour day with an additional computer system is $8\lambda.\ W_q'$

$$= 8 \times 8 \times \frac{1}{30} \text{ hrs.} = 128 \text{ minutes.}$$

The total goodwill cost with an additional computer system

$$= \text{Rs. } 128 \times \frac{12}{100} = \text{Rs. } 15{\cdot}36.$$

Hence reduction in goodwill cost with the installation of a computer system

$$= \text{Rs. } 76{\cdot}80 - \text{Rs. } 15{\cdot}36 = \text{Rs. } 61{\cdot}44.$$

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61·44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11·44 per day. It is, therefore, worthwhile to instal a computer.

Example 21. *In the production shop of a company, the breakdown of the machines is found to be Poisson distributed with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairmen. One of the repairmen is slow but cheap the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the broken down machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour and will repair machines exponentially at an average rate of 6 per hour, which repairman should be hired ?* **[AIMS (BE Ind.) Bangl. 2002; Gujarat (M.B.A.) 97]**

Solution. Here we compare the total expected hourly cost for both the repairmen which would equal the total wages paid plus the cost due to machine breakdown (*i.e.*, for the non-productive machine hours).

Cost of non-productive time

= Average number of machines in the system × Cost of idle machine hour

$$= L_s \times (\text{Rs. } 40/\text{hour}) = \frac{\lambda}{\mu-\lambda} \times 40.$$

For *slow-cheap repairman* :

$\lambda = 3$ machines per hour, $\mu = 4$ machines per hour

$$\therefore \quad L_s = \frac{3}{4-3} = 3 \text{ machines.}$$

Cost of non-productive machine time = 40×3 = Rs. 120.

$\therefore$ Total cost of slow but cheap repairman = Rs. 40×3 + Rs. 20 = Rs. 140.

For *fast-expensive repairman* :

$\lambda = 3$ machines per hour

$\mu = 6$ machines per hour

$L_s = \frac{3}{6-3} = 1$ machine.

Thus, the cost of non-productive machine time

$= 40 \times 1 =$ Rs. 40.

$\therefore$ Total cost of fast but expensive repairman

$=$ Rs. 40×1 + Rs. 30 = Rs. 70.

Obviously, the *fast repairman* should be employed by the company.

Example 22. *In a factory, the machine breakdown on an average rate is 10 machines per hour. The idle time cost of a machine is estimated to be Rs. 20 per hour. The factory works 8 hours a day. The factory manager is considering 2 mechanics for repairing the machines. The first mechanic A takes about 5 minutes on an average to repair a machine and demands wages Rs. 10 per hour. The second mechanic B takes about 4 minutes in repairing a machine and demands wages at the rate of Rs. 15 per hour. Assuming that the rate of machine breakdown is Poisson-distributed and the repair rate is exponentially distributed, which of the two mechanics should be engaged?* **[Delhi (M. Com.) 97]**

Solution. Here we shall compare the expected daily cost *viz.*, total wages paid plus cost due to machine breakdown for both the repairman.

Total wages for fast repairman = Hourly rate $\times$ No. of hours

$= 10 \times 8 =$ Rs. 80

Total wages for slow repairman = $15 \times 8 =$ Rs. 120.

Cost of non-productive time

= (Average number of machines in the system) $\times$ (Cost of idle machine hour) $\times$ (No. of hours)

$$= \frac{\lambda}{\mu - \lambda} \times 20 \times 8.$$

For *fast repairman* :

$\lambda = 10$ machines per hour, $\mu = 12$ machines per hour.

$$\therefore \text{ Total cost } = 80 + \frac{10}{(12-10)} \times 20 \times 8 = \text{Rs. } 880.$$

For *slow repairman* :

$\lambda = 10$ machines per hour, $\mu = 12$ machines per hour.

$$\therefore \text{ Total cost } = 120 + \frac{10}{(15-10)} \times 20 \times 8 = \text{Rs. } 440.$$

Obviously, the fast repairman should be employed by the company.

Example 23. *A firm has several machines and wants to instal its own service facility for the repair of its machines. The average breakdown rate of the machines is 3 per day. The repair time has exponential distribution. The loss incurred due to the lost time of an inoperative machine is Rs. 40 per day. There are two repair facilities available. Facility X has an installation cost of Rs. 20,000 and facility Y costs Rs. 40,000. The total labour cost per year for the two facilities is Rs. 5,000 and Rs. 8,000 respectively. Facility X can repair 4 machines daily while facility Y can repair 5 machines daily. The life span of both the facilities is 4 years. Which facility shoud be installed?* **[Kurukshetra (M.B.A.) Nov. 96]**

Solution. We shall compare the total annual cost of the two facilities by using the relation :

Total annual cost $= \frac{1}{4}$ (cost investment expenditure) + (annual labour cost) + (annual cost of lost revenue due to down machines).

Facility X : Annual capital cost $= \frac{1}{4}(20{,}000) =$ Rs. 5,000

Annual labour cost = Rs. 5,000

Expected number of customers in the system

$$= \frac{\lambda}{\mu - \lambda} = \frac{3}{4-3} = 3 \text{ per day.}$$

The daily cost of lost time $= 3 \times 40 =$ Rs. 120 per day

Annual cost of lost machine time $= 365 \times 120 =$ Rs. 43,800.

Total annual cost for facility X $= 5{,}000 + 5{,}000 + 43{,}800 =$ Rs. 53,800.

Facility Y: Annual capital cost $= \frac{1}{4}(40{,}000) =$ Rs. 10,000

Annual labour cost = Rs. 8,000.

Expected number of customers in the system $= \frac{3}{5-3} = \frac{3}{2}$ per day.

The daily cost of lost time $=$ Rs. $40 \times \frac{3}{2} =$ Rs. 60.

Annual cost of lost machine-time $= 365 \times 60 =$ Rs. 21,900.

Total annual cost for facility Y = Rs. 39,900.

Hence, facility Y should be preferred.

Example 24. *A tax consulting firm has four service stations (counters) in its office to receive people who have problems and complaints about their income, wealth and sales taxes. Arrivals average 100 persons in a 10-hour service day. Each tax adviser spends an irregular amount of time servicing the arrivals which have been found to have an exponential distribution. The average service time is 20 minutes. Calculate :*

(i) the average number of customers in the system,

(ii) average number of customers waiting to be serviced,

(iii) average time a customer spends in the system,

(iv) average waiting time for a customer,

(v) the probability that a customer has to wait before he gets service. **[Virbhadrah 2000; Punjab (M.B.A.) 98]**

Solution. Here we are given :

$$\lambda = 10/\text{hour}, \mu = 3/\text{hour}, k = 4 \text{ and } \rho = \frac{\lambda}{\mu} = \frac{10}{12}$$

Probability of no customer in the system is :

$$P_0 = \left[\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{\{1-(\lambda/k\mu)\}} \right]^{-1}$$

$$= \left[1 + \frac{10}{3} + \frac{1}{2}\left(\frac{10}{3}\right)^2 + \frac{1}{6}\left(\frac{10}{3}\right)^3 + \frac{1}{24}\left(\frac{10}{3}\right)^4 \frac{1}{\{1-(10/12)\}} \right]^{-1}$$

$$= 0{\cdot}0208$$

(i) Average number of customers in the system is :

$$L_s = L_q + \frac{\lambda}{\mu} = \left[\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \frac{\lambda\mu}{(k\mu-\lambda)^2} \right] P_0 + \frac{\lambda}{\mu}$$

$$= \left[\frac{1}{3!}\left(\frac{10}{3}\right)^4 \frac{30}{(12-10)^2} \right] \times 0{\cdot}0208 + \frac{10}{3} = 6{\cdot}567$$

(ii) Average queue length is given by :

$$L_q = L_s - \frac{\lambda}{\mu} = .6{\cdot}567 - \frac{10}{3} = 3{\cdot}234 \text{ customers}$$

(iii) Average time a customer spends in the system is :

$$W_s = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{3{\cdot}234}{10} + \frac{1}{3} = 0{\cdot}6567 \text{ hour}$$

(iv) Average time a customer waits for service in the queue is given by :

$$W_q = \frac{L_q}{\lambda} = \frac{3{\cdot}234}{10} = 0{\cdot}3234 \text{ hour}$$

(v) Probability that a customer has to wait is :

$$P(n \geq k) = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{[1-(\lambda/k\mu)]} P_0$$

$$= \frac{1}{4!}\left(\frac{10}{3}\right)^4 \frac{1}{[1-(10/12)]}(0{\cdot}0208) = 0{\cdot}618.$$

Example 25. *A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for both deposits and withdrawals is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in Poisson fashion throughout the day with mean arrival rate of 16 per hour. Withdrawers also arrive in Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits ? What could be the effect if this could be accomplished by increasing the mean service time to 3·5 minutes ?*

[A.I.M.A. (P.G. Dip. in Management), Dec. 98]

Solution. Initially, we have two independent queuing systems for withdrawers and depositors with input as Poisson distribution and service as exponential distribution.

For *withdrawers* : $\lambda = 14$/hour; $\mu = 3$/minute or 20/hour

Average waiting time in queue, $W_q = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{14}{20(20-14)} = \frac{14}{20\times 6} = \frac{7}{60} \text{ hour or 7 minutes.}$$

For depositors : $\lambda = 16$/hour; $\mu = 3$/minute or 20/hour.

Average waiting time in queue, $W_q' = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{16}{20(20-16)} = \frac{16}{20\times 4} = \frac{1}{5} \text{ hour or 12 minutes.}$$

If each teller could handle both withdrawals and deposits, we have a common queue with two servicers. The queuing system is thus with 2 service channels with $\lambda = 14 + 16 = 30$/hour and $\mu = 20$/hour.

Average waiting time of arrival in the queue is :

$$W_q = \frac{L_q}{\lambda} = \frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \frac{\mu}{(k\mu-\lambda)^2} P_0$$

where
$$P_0 = \left[\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{k\mu}{k\mu-\lambda}\right]^{-1}$$

EXAMINATION PROBLEMS (ON MODEL I)

1. Customers arrive at a box office window, being manned by a single individual, according to a *Poisson* input process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the average waiting time of a customer.

   Also determine the average number of customers in the system and average queue length.
   [**Hint.** Here $\lambda = 0.5$, $\mu = 0.67$. Use formula for W_q, L_s. and L_q.]
   [**Ans.** $W_Q = 4.5$ min/customer, $L_s = 3$, $L_q = 2.25$]

2. Arrivals of machinists at a toolcrib are considered to be *Poisson* distributed at an average rate of 6 per hour. The length of time the machinists must remain at the toolcrib is exponentially distributed with the average time being 0.05 hour.
   (a) What is the probability that a machinist arriving the toolcrib will have to wait ?
   (b) What is the average number of machinists at the toolcrib ?
   (c) The company will install a second toolcrib when conviced that a machinist would expect to have spend at least 6 minutes waiting and being serviced at the toolcrib. By how much must the flow of machinists to the toolcrib increase to justify the addition of a second toolerib ?

[**Hint.** Here $\lambda = \frac{6}{60}$, $\mu = \frac{1}{05 \times 60} = \frac{1}{3}$. To find W_s proceed as solved ***Example* 3**.]

[**Ans.** (a) $P(W > 0) = \frac{3}{10}$, (b) $(L \mid L > 0) = 1.43$, (c) $\lambda' - \lambda = \frac{2}{9} - \frac{1}{10} = \frac{11}{90}$.]

3. Consider a self service store with one cashier. Assume *Poisson* arrivals and exponential service times. Suppose that 9 customers arrive on the average every 5 minutes and the cashier can serve 10 in 5 minutes. Find :
(*i*) the average number of customers queueing for service.,
(*ii*) the probability of having more than 10 customers in the system,
(*iii*) the probability that a customer has to queue for more than 2 minutes.
If the service can be speeded up to 12 in 5 minutes by using a different cash register, what will be the effect on the quantities in (*i*), (*ii*) and (*iii*).

[**Hint.** Case I. $\lambda = 9/5$ per min., $\mu = 10/5$ per min. $L_s = 3$, $P(\geq 10) = (0.9)^{10}$, $P(W > 2) = 0.67$.
Case II. $\mu = 12/5$ (instead of 10/5), $\mu = 9/5$.
$L_s = 3$, $P(\geq 10) = (0.75)^{10}$, $P(W > 2) = 0.30$.]

[**Ans.** Since the average number of customers is reduced to 3 and the probability that a customer has to wait for more than 2 minutes is also reduced to 0.30, the case II will be preferable].

4. In a bank there is only one window, a solitary employee performs all the service required and the window remains continuously open from 7.00 (a.m.) to 1.00 (p.m.). It has been discovered that the average number of clients is 54 during the day and that the average service time is of five minutes per person. Calculate :
(*i*) average number of clients in the system (including the one being served)
(*ii*) the average number of clients in the waiting line (excluding the one being served), and
(*iii*) the average waiting time.

[**Hint.** $\lambda = 54/6 \times 60$, $\mu = 11/5$. Use formulae for L_s, L_q, W_s.]

[**Ans.** $L_s = 3$, $L_q = 2.25$, $W_s = 20$ min. per client.]

5. A ticket issuing office is being manned by a single server. Customers arrive to purchase tickets according to a Poisson process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the value of P_n, L_s and W_s, where L_s and W_s denote the expected line length and waiting time in the system respectively.]

[**Hint.** $\lambda = 30/60 = 1/2$, $\mu = 60/90 = 2/3$. Use formulae for P_n, L_s, W_s]

[**Ans.** $P_n = \frac{1}{4}\left(\frac{1}{4}\right)^n$, $L_s = 3$ customers, $W_s = 6$ min.]

6. An airline has one reservation clerk on duty at a time. He handles in formation about flight schedules and makes reservations. All calls to the airline are answered by an operator. If a caller requests information or reservation, the operator transfers that call to the reservation clerk. If the clerk is busy, the operator asks the caller to wait. When the clerk becomes free the operator transfers to him the call of the person who has been waiting for the longest.
Assume that arrivals and services follow *Poisson* and exponential distributions respectively. Calls arrive at a rate of ten per hour, and the reservation clerk can service a call in four minutes on the average.
(*i*) What is the average number of calls waiting to be connected to the reservation clerk ?
(*ii*) What is the average time a caller must wait before reaching the reservation clerk.
(*iii*) What is the average time for a caller to complete a call (*i.e.* waiting time plus service time) ?

[**Hint.** $\lambda = 10/60$, $\mu = 1/4$. Use formulate for L_q, W_q, W_s]

[**Ans.** $L_q = 1.33$, $W_q = 8$ min, $W_s = 12$ min.]

7. At a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes, if there is just one phone.
(*i*) What is expected number of callers in the booth at any time,
(*ii*) For what proportion of time in the booth expected to be idle.

[**Hint.** $\lambda = 1/4$ arrival/min. $\mu = 1/3$ service/min. Use formula for L_s and $1 - \rho$.]

[**Ans.** (i) $L_s = 3$ callers, (ii) 1/4.]

8. Weavers in a *Textile* Mill arrive at a *Department Store Room* to obtain spare parts needed for keeping the looms running. The store is manned by one attendant. The average arrival rate of weavers per hour is 10 and service rate per hour is 12. Both arrival and service rate follow *Poisson* process. Determine :
(*i*) Average length of waiting line.
(ii) Averge time a machine spends in the system.
(*iii*) Percentage idle time of *Department Store Room* (attendant).

[**Ans.** (i) 6 wevers, (ii) 30 min., (ii)16.67%]

9. Consider a box office ticket window being manned by single server. Customers arrive to purchase tickets according to a *Poisson* input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds.Determine the following : **[JNTU (MCA III) 2004, (B. Tech.) 2003]**
(*i*) Fraction of the time the server is busy. (*ii*) The average number of customer queueing for service.
(*iii*) The probability of having more than 10 customers in the system.
(*iv*) The probability that the customer has to queue for more than 3 minutes.

10. A repair shop attended by a single mechanic has an average of four customers a hour who bring small appliances for repair. The mechanic inspects them for defects and quite often can fix them right way or otherwise render a diagnosis. This takes him six minutes, on the average. Arrivals are *Poisson* and service time has the exponential distribution. You are required to:

(*i*) Find the proportion of time which the shop is empty, (*ii*) Find the probability of finding at least q customers in a shop, (*iii*) What is the average number of customers in the system ?
(*iv*) Find the average time spend, including service.

11. People arrive at a theater ticket booth in a *Poisson* distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate : (*i*) The mean number in the waiting line, (*ii*) The mean waiting time, (*iii*) The per cent of time an arrival can walk right in without having to wait. **[M.C.A. (May) 2000; C.A. (June) 91]**

12. Trucks arrive at the truck dock of a whole sale concern in a *Poisson* manner at 8 per hour. Service time distribution is approximated by negative exponential process with an average 5 minutes. Calculate :
(*i*) The number in waiting line, (*ii*) The waiting time
(*iii*) The mean number in the system (*iv*) The probability of having 6 trucks in the system.

13. In a *Bhawan Cafeteria* it was observed that there is only one bearer who takes exactly 4 minutes to serve a cup of coffee once the order has been placed with him. If the students arrive in a cafeteria at an average rate of 10 per hour, how much time one is expected to spend waiting for his turn to place the order ? **[Meerut (Maths.) 96]**

14. (a) Explain the constituents fo a single channel model.
(b) Customers arrive at the *First Class Ticket Counter* of a theatre at a rate of 12 per hour. There is one clerk serving the customers at the rate of 30 per hour :
(*i*) What is the probability that there is no customer in counter (*i.e.*, the system is idle) ?
(*ii*) What is the probability that there are more than 2 customers in the counter ?
(*iii*) What is the probability that there is no customer waiting to be served ?
(*iv*) What is the probability that a customer is being served and nobody is waiting ? **[C.A. (Nov.) 90]**

[**Hint.** Here $\lambda = 12$ per hour, $\mu = 30$ per hour, $\rho = 2/5$]

[**Ans.** (i) $P_0 = 0.6$, (ii) $P(N > 2) = \rho^3 = 0.064$, (iii) $P_0 + P_1 = 0.84$, (iv) $P_1 = 0.24$]

15. A computer manufacturing concen has sold a new brand of mini-computers to ten different organizations (one each) in a locality. The concen has employed one full time engineer to look into the complaints of malfunctioning. If the computers that malfunction arrive at the concen in a *Poisson* manner with rate 0.5 per unit time and the repair time of any particular machine is exponentially distributed with mean repair time of any particular machine is exponentially distributed with mean repair time equal to 0.5 unit, determine the steady state probability of the number of mini-computers queueing up for repairs.

16. A hospital is studying the proposal to reorganise its emergency service facility. The present arrival rate at the emergency service is 1 call every 15 minutes and the service rate is 1 call every 10 minutes. Current cost of service is Rs. 100 per hour. Each delay in service is Rs. 125. If the proposal is accepted, the service rate will become 1 call every 6 minutes. Can the organisation be justified on a strictly cost basis it the proposal increases the cost of the service by 50 %. **[Garhwal M.Sc. (Math.) 94]**

17. Patients arrive in a *Dental OPD* of general hospital in a *Poisson* manner at an average rate of 6 per hour. The doctor on average can attend to 8 patients per hour. Assuming that the service time distribution for the doctor is exponential, find: (i) Average number of patients waiting in the queue. (ii) Average time spent by a patient in the dental OPD. **[Delhi (MBA) Dec. 94]**

18. (a) Patients arrive at a clinic according to Poisson distribution at the rate of 30 patients per hour. Estimation time per patient is exponential with mean rate 20 per hour. If capacity of the clinic is unlimited, find the prob. that an arriving patient will not wait deriving the formula you use. **[Agra 98; Meerut (Math.) 98 BP]**
[**Hint.** Proceed as Ex. 8, page 242]

(b) Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. the waiting room does not accomodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.
(i) Find the effective arrival rate at the clinic.
(ii) What is the probability that an arriving patient will not wait ?
(iii) What is the expected waiting time until a patient is discharged from the clinic ? **[JNTU (B. Tech.) 2003]**

19. A maintenance service facility has Poisson arrival rates, negative exponential service times, and operates on a first-come first-served queue discipline. Breakdowns occur on an average of three per day with a range of zero to eight. The maintenance crew can service on an average six machines per day with a range from zero to seven. Find the
(i) Utilisation factor of the service facility,
(ii) mean time in the system,
(iii) mean number in the system in breakdown or repair,
(iv) mean waiting time in the queue,
(v) probability of finding two machines in the system.
(vi) expected number in the queue. **[M.G. (M.B.A.) Dec. 98]**

[**Ans.** (i) $\rho = 50\%$ (ii) $W_s = \frac{1}{3}$ per day (iii) $L_s = 1$ machine, (iv) $W_1 = \frac{1}{6}$ per day, (v) $P_2 = 0{\cdot}125$, (vi) $L_q = \frac{1}{2}$ machine]

20. A bank has one drive-in-counter. It is estimated that cars arrive according to Poisson distribution at the rate of 2 every 5 minutes and that there is enough space to accommodate a line of 10 cars. Other arriving cars can wait outside this space, if necessary. It takes 1·5 minutes on an average to serve a customer, but the service time actually varies according to an exponential distribution. You are required to find :
(i) the proportion of time the facility remains idle;
(ii) the expected number of customers waiting but currently not being served at a particular point of time;
(iii) the expected time a customer spends in the system, and
(iv) the probability that the waiting line will exceed the capacity of the space leading to the drive-in counter. **[C.A. (May) 97]**

[**Hint.** $\lambda = \frac{2 \times 60}{5}$, $\mu = \frac{60}{1.5}$

Ans. (i) $P_0 = 0.4$, *i.e.*, 40% (ii) $L_q = 0.9$, (iii) $W_s = 3.75$ minutes (iv) $P(n \geq 11) = (\lambda/\mu)^{11} = 0.036$]

21. Arrivals of customers to a payment counter (only one) in a bank follow Poisson distribution with an average of 10 per hour. The service time follows negative exponential distribution with an average of 4 minutes.
(i) What is the average number of customers in the queue ?
(ii) The bank will open one more counter when the waiting time of a customer is at least 10 minutes. By how much the flow of arrivals should increase in order to justify the second counter ? **[Jammu (M.B.A.) 97]**

[**Hint.** $\lambda = 10$/hour and $\mu = 15$/hour.

Ans. (i) $L_q = 4/3$ (ii) $W_q' = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{10}{60} = \frac{\lambda'}{15(15 - \lambda')}$ or $\lambda' = 10.7$.

Thus arrival rate should increase by 0·7 hour.]

22. A refinery distributes its products by trucks loaded at the loading dock. There are both company trucks and independent distributors' trucks. The independent distributors complained that sometimes they must wait in line and thus lose money paying for truck and driver that is only waiting. They have asked the refinery either to put in a second loading dock or to discount price equivalent to the waiting time. Extra loading dock costs Rs. 100 per day wheres the waiting time for the independent firms costs Rs. 25 per hour.

The following data of arrival and service at the dock are available and they are Poisson distributed :

Average arrival rate of all trucks = 2 per hour.
Average service rate = 3 per hour.

Thirty per cent of all trucks are from independent distributors.
(i) What is the expected cost of waiting time of the independent distributors ?
(ii) Is it advantageous to decide in favour of a second loading dock to ward off the complaints ? **[Bombay (M.M.S.) 95]**

23. A single crew is provided for unloading and/or loading each truck that arrive at the loading deck of a warehouse. These trucks arrive according to a Poisson input process at a mean rate of one hour. The time required by a crew to unload and/or laod a truck has an exponential distribution (regardless of the crew size). The expected time required by a one-man crew would be two hours.

The cost of providing each additional member of the crew is Rs. 10 per hour. The cost that is attributable to having a truck not in use (*i.e.*, a truck standing at the loading deck) is estimated to be Rs. 40 per hour.

Assume that the mean service rate of the crew is proportional to its size. What should be the size in order to minimize the expected total cost per hour ? **[Delhi (M.B.A.) 96]**

24. A factory operates for 8 hours everyday and has 240 working days in the year. It buys a large number of small machines which can be serviced by its maintenance engineer at a cost of Rs. 5 per hour for the labour and spare parts. The machines can, alternatively, be serviced by the supplier at an annual contact price of Rs. 25,000 including labour and spare parts needed. The supplier undertakes to send a repairman as soon as a call is made but in no case more than one repairman is sent. The service time of the maintenance engineer and the supplier's repairman are both exponentially distributed with respective means of 1·7 and 1·5 days. The machine breakdowns occur randomly and follow Poisson distribution, with an average of 2 in 5 days. Each hour that a machine is out of order, it costs the company Rs. 10. Which servicing alternative would you advise it to opt for ? **[Allahabad (M.B.A.) 97]**

[**Hint.** *Alternative* 1 : *Sarviced by company's Engineer*

$\lambda = 2/5$ mach/day, $\mu = 1/1.7$ mach/day

Total cost = cost of machine x + cost of labour and spare parts

$$= \left(\frac{\lambda}{\mu - \lambda}\right)(240 \times 8) \times 10 + \left(\frac{\lambda}{\mu - \lambda}\right)(240 \times 8) \times 5$$

= Rs. 61,200

Alternative 2. Mantenance by supplier :

$$\lambda = \frac{2}{3} \text{ mach/day}, \ \mu = \frac{1}{1.5} \text{ mach/day}$$

$$\text{Total cost} = \frac{3}{2} \times 1920 \times 10 + 25{,}000 = \text{Rs. } 53{,}800$$

$\therefore$ second alternative is the best one.]

25. A car park has space to accomodate 40 cars. The arrival of cars is Poisson at a mean rate of 2 per minute. The length of time each car spends in the car rank has negative exponetial distribution with mean of 30 minutes ?

(a) What is the probability of a newly arriving customer finding the car park full ?

(b) How many cars are in the car park on average ?

(c) What is the probability of having zero cars in the car park space. [JNTU (B. Tech.) 2003]

26. A company currently has two tools cribs, each having a single clerk, in its manufacturing area. One tool crib handles only the tools for the heavy machinery, while the second one handles all other tools. It is observed that for each tool crib the arrivals follow a Poisson distribution with a mean of 20 per hour, and the service time distribution is negative exponential with a mean of 2 minutes.

The tool manager feels that, if tool cribs are combined in such a way that either clerk can handle any kind of tool as demand arises, the system would be more efficient and the waiting problem could be reduced to some extent. It is believed that the mean arrival rate at the two tool cribs will be 40 per hour; while the service time will remain unchanged. Compare the existing system with the one proposed with respect to the total expected number of machines at the tool crib(s), the expected waiting time including service time for each mechanic and probability that he hasa to wait for service. [Delhi (M.B.A.) March 99]

27. The men's department of a large store employs one tailor for customer fittings. The number of customers requiring fittings appears to follow a poisson distribution with mean arrival rate 24 per hour. Customers are fitted on a first-come, first-served basis, and they are always willing to wait for the tailors service, because alterations are free. The time it takes to fit a customer appears to be exponentially distributed, with a mean of 2 min.

(i) What is the average number of customers in the fitting room ?

(ii) How much time a customer is expected to spend in the fitting room ?

(iii) What percentage of the time is the tailor idle ? [VTU (BE Mech.) 2002]

28. Problems arrive at a computing center in Poisson fashion with a mean arrival rate of 25% per hour. The average computing job requires 2 minutes of terminal of time. Calculate the following :

(a) average no. of problems waiting for the computer.

(b) the percentage of times on arrival can walk right in without having to wait. [JNTU (MCA III) 2004]

29. A firm is engaged in both shipping and receiving activities. The management is always interested in improving the efficiency by new innovations in loading and unloading procedures. The arrival distribution of trucks is found to be Poission with arrival rate of two trucks per hour. The service time distribution is exponential with unloading rate of three trucks per hour. Find the following :

(a) Average no. of trucks in waiting time (b) Average waiting time of trucks in line.

(c) The prob. that the loading and unloading dock and workers will be idle.

(d) What reductions in waiting time are possible if loading and unloading is standarized ?

(d) What reductions are possible if lift trucks are used. [JNTU (Mech. & Prod.) Main 2004]

10.13. Model II (A). General Erlang Queueing Model (Birth-Death Process) [Agra 98]

(a) To obtain the system of steady state equations

Let

$$\left.\begin{array}{l}\text{arrival rate } \lambda = \lambda_n \\ \text{service tate } \mu = \mu_n\end{array}\right\} \quad \text{[depending upon } n]$$

Then, by the same arguments as for equations (10.51) and (10.53),

$$P_n(t+\Delta t) = P_n(t)\,[1-(\lambda_n+\mu_n)\,\Delta t] + P_{n-1}(t)\,\lambda_{n-1}\Delta t + P_{n+1}(t)\,\mu_{n+1}\,\Delta t + O(\Delta t),\ n>0; \quad ...(10.73)$$

and $$P_0(t+\Delta t) = P_0(t)\,[1-\lambda_0\,\Delta t] + P_1(t)\,\mu_1\,\Delta t + O(\Delta t),\ n=0. \quad ...(10.74)$$

Now dividing (10.73) and (10.74) by Δt, taking limits as $\Delta t \to 0$ and following the same procedure as in Model I, obtain

$$\frac{dP_n(t)}{dt} = -(\lambda_n+\mu_n)\,P_n(t) + \lambda_{n-1}\,P_{n-1}(t) + \mu_{n+1}\,P_{n+1}(t) \quad ...(10.75)$$

and $$\frac{dP_0(t)}{dt} = -\lambda_0\,P_o(t) + \mu_1\,P_1(t) \text{ respectively.} \quad ...(10.76)$$

The equations (10.75) and (10.76) are differential-difference equations which could be solved if a set of initial values $P_0(0)$, $P_1(0)$, ..., is given. Such a system of equations can be solved if the time dependent solution is required. But, for many problems it suffices to look at the steady state solution.

In the case of steady state,

$$P_n'(t) = 0 \text{ and } P_0'(t) = 0.$$

So the equations (10.75) and (10.76) become,

$$0 = -(\lambda_n+\mu_n)\,P_n + \lambda_{n-1}\,P_{n-1} + \mu_{n+1}\,P_{n+1},\ n>0 \quad ...(10.77)$$

and $$0 = -\lambda_0\,P_0 + \mu_1\,P_1,\ n=0. \quad ...(10.78)$$

The equations (10.77) and (10.78) constitute the system of steady state difference equations for this model. **[Agra 98]**

(b) To solve the system of difference equations

Since $P_0 = P_0$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \qquad \text{[from equation (10.78)]}$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0 \qquad \text{[from letting } n = 1 \text{ in equation (10.77) and substituting for } P_1]$$

$$P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \qquad \text{[putting } n = 2 \text{ in equation (10.77)]}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$P_n = \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_1} P_0 \quad [\text{for } n \geq 1] \qquad \ldots(10.79)$$

Now, in order to find P_0, use the fact that

$$\sum_{n=0}^{\infty} P_n = 1. \text{ or } P_0 + P_1 + P_2 + \ldots \infty = 1 \text{ or } P_0 \left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty\right] = 1 \text{ or } P_0 = 1/S,$$

where

$$S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty. \qquad \ldots(10.80)$$

Note. The series $S = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \ldots \infty$ is summable and meaningful only when it is convergent.

The result obtained above is a general one and by suitably defining μ_n and λ_n many interesting cases could be studied. Now three particular cases may arise :

Case 1. ($\lambda_n = \lambda$, $\mu_n = \mu$)

In this case, the series S becomes $\quad S = 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots \infty = \frac{1}{1 - \lambda/\mu} \qquad$ (when $\lambda/\mu < 1$)

Therefore, from equations (10.80) and (10.79),

$$P_0 = \frac{1}{S} = 1 - \frac{\lambda}{\mu} \quad \text{and} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Here, it is observed that this is exactly the case of Model I.

Case 2. $\left(\lambda_n = \frac{\lambda}{n+1}, \mu = \mu\right)$

The case, in which the arrival rate λ_n depends upon n inversely and the rate of service μ_n is independent of n, is called the case of *"Queue with Discouragement"*.

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{2.3\mu^3} + \ldots \infty = 1 + \rho + \frac{\rho^2}{2!} + \frac{\rho^3}{3!} + \ldots \infty = e^{\rho} (\rho = \lambda/\mu)$$

The equation (10.80) gives $P_0 = 1/S = e^{-\rho}$.

Also,

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho\, e^{-\rho},$$

$$P_2 = \frac{\lambda^2}{2\mu^2} P_0 = \frac{\rho^2 e^{-\rho}}{2!},$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$P_n = \frac{\lambda^n}{n!\,\mu^n} P_0 = \frac{\rho^n e^{-\rho}}{n!} \text{ for all } n = 0, 1, \ldots, \infty.$$

It is observed in this case that P_n follows the Poisson distribution, where $\lambda/\mu = \rho$ is constant, however $\rho > 1$ or $\rho < 1$ but must be finite. Since, the series S is convergent and hence summable in both the cases.

Case 3. ($\lambda_n = \lambda$ **and** $\mu = n\mu$),*i.e.*, **the case of infinite number of stations.**

In this case, the arrival rate λ_n does not depend upon n, but the service rate μ_n increases as n increases. Here, assume that there are infinite (variable) number of service stations. The word '*infinite*' means that the service stations are available to each arrival. But it does not mean that all the infinite service stations will remain busy every time. In other words, it means that if n customers arrive, then n service stations will be available for all $n = 0, 1, 2, \ldots\infty$. Obviously, no queue will form in this case because each arrival will immediately enter the service facility. For example, in everyday life, it is observed that the telephone (service stations) are always available to all the arriving persons.

In this case, the series S becomes

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2.1\mu^2} + \frac{\lambda^3}{3.2.1\mu^3} + \ldots\infty = 1 + \rho + \frac{\rho^2}{2\,!} + \frac{\rho^3}{3\,!} + \ldots\infty = e^{\rho}, \text{ where } \rho = \lambda/\mu.$$

Therefore, $\quad P_0 = e^{-\rho}$ and $P_n = e^{-\rho}\rho^n/n\,!$

which again follows the *Poisson* distribution law.

10.13-1. Model II (B). (*M* | *M* | 1) : (∞ | *SIRO*)

This model is actually the same as *Model* I, except that the service discipline follows the *Service In Random Order* (*SIRO*) rule in place of *FCFS*-rule. Since the derivation of P_n in *Model* I does not depend on any specific queue discipline, we must have

$$P_n = (1 - \rho)\,\rho^n, n \geq 0$$

for the *SIRO*-rule case also.

Consequently, the average number of customers in the system (L_s) will remain the same, whether the queue discipline follows the *SIRO*-rule or *FCFS*-rule. In fact, L_s will not change provided P_n remains unchanged for any queue discipline. Thus $W_s = L_s/\lambda$ under the *SIRO*-rule is also same as under the *FCFS*-rule and it is given by $W_s = 1/(\mu - \lambda)$.

Moreover, this result can be extended to *any queue discipline* so long as P_n remains unchanged. In particular, the result is applicable to the three most common queue disciplines : *FCFS, LCFS* and *SIRO*. These queue disciplines differ only in the distribution of waiting time where the probabilities of long and short waiting times change depending upon the queue discipline used. So we can use the *GD* (*General Discipline*) to represent *FCFS, LCFS* and *SIRO*, whenever the waiting time distribution is not required.

Q. 1. Deduce the difference equations for the queueing model M | M | 1 : (FCFS/∞/∞) with arrival and service rates dependent on system size. Obtain steady state solution. Deduce also the solution for the following special cases :
(i) Queues with discouragement, (ii) Queues with ample servers. **[Delhi MA/M.Sc. (OR.) 92, 90]**

2. Derive differential-difference equations for a generalized birth-death queueing model. Obtain steady-state distribution of the system size. **[Delhi MA/M.Sc (Stats.) 95]**

10.13–2. Illustrative Examples on Model II

Example 26. *Problems arrive at a computing centre in a Poisson fashion at an average rate of five per day. The rules of the computing centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of ⅓ day, and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line.* **[Rohil. 92]**

Solution. Here $\lambda = 5$ problems/day

$\mu = 3$ problems/day (mean service rate with one unsolved problem)

Then, the expected number of persons working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} nP_n\,, \text{ where } P_n = \frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \text{ [see Case II]}$$

$$= \sum_{n=0}^{\infty} n\frac{1}{n\,!}\left(\frac{\lambda}{\mu}\right)^n e^{-\lambda/\mu} \qquad = e^{-\lambda/\mu}\sum_{n=0}^{\infty}\frac{n}{n\,!}\cdot\left(\frac{\lambda}{\mu}\right)^n$$

$$= e^{-\lambda/\mu}\left\{\frac{\lambda}{\mu}+\frac{2}{2!}\left(\frac{\lambda}{\mu}\right)^2+\frac{3}{3!}\left(\frac{\lambda}{\mu}\right)^3+\ldots\infty\right\}$$
$$= e^{-\lambda/\mu}\cdot\frac{\lambda}{\mu}\left\{1+\frac{\lambda}{\mu}+\frac{1}{2!}\cdot\left(\frac{\lambda}{\mu}\right)^2+\ldots\right\}$$
$$= e^{-\lambda/\mu}\cdot\frac{\lambda}{\mu}e^{\lambda/\mu} = \lambda/\mu\,.$$

Substituting the values for λ and μ, $L_S = 5/3$ persons.

Now, the average solving time which is inversely proportional to the number of people working on the problem is : $1/5$ day/problem.

Therefore, expected time for a person entering the line is

$$= 1/5 \times L_S \text{ days} = 1/5 \times 5/3 \times 24 \text{ hours} = 8 \text{ hours.}$$ **Ans.**

Example 27. *A shipping company has a single unloading berth with ships arriving in Poisson fashion at an average rate of three per day. The unloading time distribution for a ship with the unloading crews is found to be exponential with average unloading time ½ in days. The company has a large labour supply without regular working hours and to avoid long waiting lines the company has a policy of using as many unloading crews as there are ships waiting in line or being unloaded. Under these conditions,* find

(i) *the average number of unloading crews working at any time,* and

(ii) *the probability that more than four crews will be needed.*

Solution. Here,

$\lambda = 3$ ships per day

$\mu = 2$ ships per day (mean service rate with one unloading crew)

(i) Average number of unloading crews working at any specified instant is :

$$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n\frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n \left[\text{since } P_n = \frac{e^{-\lambda/\mu}}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

$= \lambda/\mu = 3/2$ crews. (*see example 26 above*) **Ans.**

(ii) The probability that more than 4 crews will be needed is the same as the probability that there are at least five ships in the system at any specified instant which is given by

$$\sum_{n=5}^{\infty} P_n = \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{4} P_n = 1-[P_0+P_1+P_2+P_3+P_4]$$
$$= 1-\left[e^{-\lambda/\mu}+\left(\frac{\lambda}{\mu}\right)\frac{e^{-\lambda/\mu}}{1!}+\left(\frac{\lambda}{\mu}\right)^2\frac{e^{-\lambda/\mu}}{2!}+\left(\frac{\lambda}{\mu}\right)^3\frac{e^{-\lambda/\mu}}{3!}+\left(\frac{\lambda}{\mu}\right)^4\frac{e^{-\lambda/\mu}}{4!}\right]$$
$$= 1-e^{-\lambda/\mu}\left[1+\frac{\lambda}{\mu}+\frac{(\lambda/\mu)^2}{2!}+\frac{(\lambda/\mu)^3}{3!}+\frac{(\lambda/\mu)^4}{4!}\right]$$

Now putting values for λ and μ, and simplifying, we get

$$\sum_{n=5}^{\infty} P_n = 0.019.$$ **Ans.**

10.14. MODEL III. $(M \mid M \mid 1) : (N \mid FCFS)$

Up to this stage, only two models are discussed in which the capacity of the system is infinite. Now consider the case where the capacity of the system is limited, say N. In fact the number of arrivals will not exceed the N in any case.

The physical interpretation for this model may be either :

(i) that there is only room (capacity) for N units in the system (as in a packing lot),

or *(ii)* that the arriving customers will *go for their service elsewhere* permanently, if the waiting line is too long ($\leq N$).

(a) To obtain steady state difference equations. The simplest way of starting this is to treat the model as a special case of *Model* II, where

$$\lambda_n = \begin{cases}\lambda\,, & n = 0, 1, 2, 3, \ldots, N-1\\ 0, & n \geq N\end{cases} \qquad \ldots(10.81)$$

and $\mu_n = \mu$ for $n = 1, 2, 3, \ldots$ ...(10.82)

Now, following the similar arguments as given for equations (10.53) and (10.51) in Model I, we obtain

$$P_0(t+\Delta t) = P_0(t)[1 - \lambda\Delta t] + P_1(t)\,\mu\Delta t + O(\Delta t), \quad \text{for } n = 0, \quad ...(10.83)$$

$$P_n(t+\Delta t) = P_n(t)[1 - (\lambda+\mu)\Delta t] + P_{n-1}(t)\,\lambda\Delta t + P_{n+1}(t)\,\mu\,\Delta t + O(\Delta t),$$
$$\text{for } n = 1, 2, \ldots, N-1, \quad ...(10.84)$$

and
$$P_N(t+\Delta t) = P_N(t)[1 - (0+\mu)\Delta t] + P_{N-1}(t)\,\lambda\,\Delta t + 0 \times \mu\,\Delta t + O(\Delta t)$$
$$= P_N(t)[1 - \mu\Delta t] + P_{N-1}(t)\,\lambda\Delta t + O(\Delta t) \quad \text{for } n = N,\ P_{N+1}(t) = 0,\ \lambda = 0 \quad ...(10.85)$$

Now dividing equation (10.83), (10.84), and (10.85) by Δt, and taking limit as $\Delta t \to 0$, these equations transform into

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0 \quad ...(10.83a)$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \quad \text{for } n = 1, 2, \ldots, N-1, \quad ...(10.84a)$$

and
$$P_N'(t) = -\mu P_N(t) + \lambda P_{N-1}(t) \quad \text{for } n = N. \quad ...(10.85a)$$

In the case of steady state, when $t \to \infty$, $P_n(t) \to P_n$ (independent of t) and hence $P_n'(t) \to 0$. So the system of steady state difference equations is given by

$$0 = -\lambda P_0 + \mu P_1, \text{ for } n = 0 \quad ...(10.83b)$$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad \text{for } n = 1, 2, \ldots, N-1, \quad ...(10.84b)$$

$$0 = -\mu P_N + \lambda P_{N-1}, \quad \text{for } n = N. \quad ...(10.85b)$$

(b) To solve the system of difference equations (10.83b), (10.84b) and (10.85b).

Here

$P_0 = P_0$ (initially)

$P_1 = \dfrac{\lambda}{\mu} P_0$ [from (10.83b)]

$P_2 = \left(\dfrac{\lambda}{\mu}\right)^2 P_0$ [put $n = 1$ in (10.84b) and substitute value of P_1].

Similarly,

$P_3 = \left(\dfrac{\lambda}{\mu}\right)^3 P_0$

... ...

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0,\ n < N \quad ...(10.86)$$

$P_N = \left(\dfrac{\lambda}{\mu}\right)^N P_0,\ n = N$ (because $P_N = (\lambda/\mu)\,P_{N-1}$ and P_{N-1} follows the rule for which $n = 1, 2, \ldots, N-1$).

$P_{N+1} = 0,\ n > N.$

Now, in order to find P_0, use the fact that:

$$\sum_{n=0}^{N} P_n = 1$$

or
$$P_0[1 + (\lambda/\mu) + (\lambda/\mu)^2 + \ldots (\lambda/\mu)^N] = 1$$

or
$$P_0\left[\frac{1-\rho^{N+1}}{1-\rho}\right] = 1, \text{ where } \rho = \lambda/\mu$$

or
$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}. \quad ...(10.87)$$

Substituting the value of P_0 in (10.86),

$$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n, \text{ for } n = 0, 1, 2, \ldots, N. \quad ...(10.88)$$

Thus, the result (10.87) and (10.88) give the required solution for this model which do not require $\lambda < \mu$. That is, in this case, ρ may be greater than 1 also.

(c) Measures of Model III.

(i) $L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$

or $$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n\rho^n = P_0 \sum_{n=0}^{N} n\,\rho^n. \qquad ...(10.89)$$

Now, four relationships (10.69), (10.70), (10.71) and (10.72) give :

(ii) $L_q = L_s - \lambda/\mu$ ...(10.90)

(iii) $W_s = L_s/\lambda$, and ...(10.91)

(iv) $W_q = W_s - 1/\mu = L_q/\lambda$. ...(10.92)

Q. 1. Obtain the steady state difference equations for the queueing model $(M \mid M \mid 1) : (N \mid FCFS)$ in usual notations and solve them for P_0 and P. Also find the mean queue length for this system.

[Meerut (Stat.) 98; Garhwal (Stat.) 92]

2. For the $(M \mid M \mid 1) : (FCFS, K)$ queueing model, show that the steady state probability, p_n is given by

$$p_n = \rho^n \frac{1-\rho}{1-\rho^{K+1}},\ 0 \le n \le K.$$

Also obtain expected number of units in the queue and system separately.

3. For the model $(M \mid M \mid 1) : (N \mid FCFS)$ where the notations have their usual meanings, find the following :

(i) The average number of customers in the system. (ii) Average queue length.

4. Explain $(M \mid M \mid 1) : (N \mid FCFS)$ system and solve it in steady state.

[Garhwal M.Sc. (Stat.) 96, 95, 93]

10.14-1. Illustrative Examples on Model III

Example 28. *In Example 5 of sec. 10.12-2, if we assume that the line capacity of yard is to admit of 9 trains only (there being 10 lines, one of which is ear marked for the shunting engine to reverse itself from the crest of the hump to the rare of the train). Calculate the following on the assumption that 30 trains, on average, are received in the yard :*

(a) *the probability that the yard is empty,*

(b) *average queue length.*

Solution. As already computed in *Example* 5, **section 10.12-2,** $\rho = 0.75$.

(a)The probability 'that the queue size is zero' is given by

$$P_0 = (1-\rho)/(1-\rho^{N+1})$$

But given that $N = 9$ so,

$$P_0 = \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.90} = 0.28.$$ **Ans.**

(b) Average queue length is given by the formula

$$L_s = \left(\frac{1-\rho}{1-\rho^{N+1}}\right) \sum_{n=0}^{N} n\rho^n$$

$$\therefore \quad L_s = \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^{9} n\,(0.75)^n = 0.28 \times 9.58 = 2.79, \text{ say 3 trains.}$$ **Ans.**

Example 29. *If for a period of 2 hours in a day (8—10 A.M.) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, then calculate for this period :*

(a) *the probability that the yard is empty,* (b) *average queue length, on the assumption that the line capacity of the yard is limited to 4 trains only.*

Solution. Here, $\rho = 36/20 = 1.8$ (which is greater than 1) and $N = 4$.

Thus, we obtain

(a) $P_0 = \frac{\rho - 1}{\rho^5 - 1} = 0.04$

(b) average queue size $= P_0 \sum_{n=0}^{4} n \rho^n$, $= .04\ [\rho + 2\rho^2 + 3\rho^3 + 4\rho^4]$,

$= .04 \times 72.0 = 2.9$, say 3 trains. **Ans.**

EXAMINATION PROBLEMS (ON MODEL III)

1. Discuss the stationary state of the queue system $(M \mid M \mid 1) : (N \mid FCFS)$.

   A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean of 5 hours. How many cars are in the car park on average and what is the probability of a newly arriving customer finding the car park full and having to park his car elsewhere ?

   [**Hint.** Here $N = 5$, $\lambda = 10/60$, $\mu = 1/2 \times 60$, $\rho = \lambda/\mu = 20$.

   Find $P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}$, $L_s = P_0 \sum_{n=0}^{N} n\rho^n$.]

2. A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full he goes to the next shop. Customers randomly arrive at an average rate $\lambda = 10$ per hour and the barber's service time is negative exponential with an average of $1/\mu = 5$ minutes per customer.
   (i) Write recurrence relations for the steady state queueing system (*FCFS*) for above.
   (ii) Determine P_0 and P_n, probability of having 0 and *n*-customers respectively in the shop.

   [**Hint.** Here $N = 10$, $\lambda = 10/60$, $\mu = 1/5$, $\rho = \lambda/\mu = 5/6$.

   Find $P_0 = \frac{1 - \rho}{1 - 'rho^{N+1}}$, $P_n = P_0\rho^n$.]

3. Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.
   (i) Find the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait ? Will he find a vacant seat in the room ?
   (iii) What is the expected waiting time until a patient is discharged from the clinic ?

   [**Hint.** Here $N = 14$, $\lambda = 30/60$, $\mu = 20/60$, $\rho = 2/3$.

   Find P_n, P_0 and $W_s = \frac{P_0}{\lambda} \sum_{n=0}^{N} n \rho^n$.] **[Meerut (MCA) 2000]**

4. Customers arrive at a one-window derive-in-counter according to a Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The car space in front of the window, including that for the serviced can accomodate a maximum of 3 cars. Other cars can wait outside this space.
   (a) What is the probability that an arriving customer can drive directly to the space in front of the window ?
   (b) What is the probability that an arriving customer will have to wait outside the indicated space ?
   (c) How long is an arriving customer expected to wait before starting service ?
   (d) How, many spaces should be provided in front of the window so that all the arriving customers can wait in front of the window at least 20% of the time.

   [JNTU (Mech. & Prod.) 2004; Meerut (MCA) 2000]

   [**Hint.** $\lambda = 10$, $\mu = 60/5 = 12$, $P_0 = 1 - (\lambda/\mu) = 1/6$, (a) $P_0 + P_1 + P_2 = [1 + \lambda/\mu + (\lambda/\mu)^2]\ P_0 = 0.42$,
   (b) $1 - (P_0 + P_1 + P_2 + P_3) = 1 - 0.42 - P_3 = 0.58 - (\lambda/\mu)^3\ P_0 + 0.48$. (c) $W_q = \lambda/\{\mu\ (\mu - \lambda)\} = 0.417$,
   (d) $P_0 + P_1 = 0.30$. Hence there should be at least one car space for waiting at least 20% of the time.]

5. A stenographer has 5 person for whom she performs stenographic work. Arrival rate is Poisson and service times are exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Cost of waiting is Rs. 8 per hour while the cost of servicing is Rs. 2.50 each. Calculate :
   (i) the average waiting time of an arrival,
   (ii) the average length of the waiting line,
   (iii) the average time which an arrival spends in the system, and
   (iv) the minimum cost service rate.

   [**Ans.** (i) 12.4 min.,
   (ii) 0.79 = one stenographer,
   (iii) 22.4 min.]

10.24. QUEUEING CONTROL

During past three decades there has been a rapidly increasing interest in the study of designing and controlling of behaviour of queueing system. Majority of queueing literature involve *prescriptive* models rather than *descriptive*. Prescriptive models are viewed as static optimization models.

Static optimization models are those in which steady-state conditions are set up for the system and some long run average criterion (such as cost and/or profit) is determined.

In static models, the configuration of the system is set once for all.

If the queueing systems depend upon time and are controlled, then these systems are known as *dynamic control systems.* Some optimization models are the mixture of *static* and *dynamic* categories. But, if the state dependent system is controlled, then it comes under *dynamic control.* Much amount of work has been done on dynamic control systems.

Objectives of Dynamic control :

Following are the objectives of dynamic control :

(*a*) ***Arrival Process control***
- (*i*) To accept or reject the control
- (*ii*) To adjust mean arrival rate
- (*iii*) Customer exercised control
- (*iv*) Self versus social optimization
- (*v*) Projection times.

(*b*) ***Service Process control***
- (*i*) Varying the number of servers
- (*ii*) Varying the service times.

Control of Queue Discipline :

There is one more branch of optimization which is named as '*control of Queue Disciplines*', Priority models, scheduling models, and allocation of customers to multi-server fall in this category.

Q. 1. Discuss the queueing model which applies to a queueing system having a single service channel, Poisson input, exponential service, assuming that there is no limit on the system capacity while the customers are served on a first come basis out basis.

2. Give essential characteristics of the queueing process. What are non-poission queues ? **[Meerut (Stat.) 95]**

3. "Queueing theory can be used effectively in determining optimal service levels." Elucidate this statement with the help of an example. **[Delhi (M.B.A.) Dec. 94]**

4. Define Queue. Write the characteristics of Queueing system. **[Bhubnashwar (IT) 2004]**

SELF-EXAMINATION PROBLEMS

1. At a one-man barber shop, customers arrive according to Poisson distribution with a mean arrival rate of 5 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following.

(i) Average number of customers in the shop and the average number of customers waiting for a hair cut.

(ii) The percentage of customers who have to wait prior of getting into the barber's chair.

(iii) The per cent of time an arrival can walk without having to wait.

[**Hint.** Here $\lambda = 1/12$ per minute, $\mu = 1/10$ per minute.]

[**Ans.** (i) $L_s = 4.8$, $L_q = 4$ (approximately)

(ii) P(queue size ≥ 1) $= \rho = 0.833$. Therefore, percentage of customers who have to wait = 83.3%

(iii) Per cent of time an arrival can walk without waiting = 100 − 83.3 = 16.7%.]

2. An overhead crane moves jobs from one machine to another and must be used every time a machine requires loading or unloading. The demand for service is random, data taken by recording the elapsed time between service calls followed as exponential distribution having a mean of a call every 30 minutes. In a similar manner, the actual service time of loading or unloading took an average of 10 minutes. If the machine time is valued at Rs. 8.50 per hour, how much does the down time cost per day ? (Assume one day = 8 working hrs.)

[**Hint.** $\lambda = 60/30$ per hour, $\mu = 60/10$ per hour. Find Down time (W_s) = 0.25 per hour. Since daily demand = 8λ = 16 calls per day, and each call requires 0.25 hour,

$\therefore$ Total cost per day = Rs. $(16 \times 0.25 \times 8.50)$.]

3. A duplicating machine maintained for office-use is used and operated by people in office who need to make copies, mostly secretaries. Since the work to be copied varies in length (number of pages of the original) and copies required, the service rate is randomly distributed, but it does approximate a Poisson having a mean service rate of 10 jobs per hour. Generally, the requirements for use are random over the entire 8-hour work day but arrive at a rate of 5 per hour. Several people have noted that a waiting line develops occasionally and have questioned the policy of maintaining only one unit. If the time of a secretary is valued at Rs. 3.50 per hour, then determine :
   (a) The per cent of time the equipment is used. (b) The per cent of time that an arrival has to wait.
   (c) Average waiting time of an arrival in the system.
   (d) The average cost due to waiting and operating the machine. **[I.C.W.A. (June) 90]**

   [**Hint.** Here $\lambda = 5$, $\mu = 10$.]
   (a) Utilization factor $(\rho) = \lambda/\mu = 1/2$. Therefore, equipment is used 50% of the time. (b) Busy period = 0.50.
   (c) $W_s = \dfrac{1}{\mu - \lambda} = 0.20$ hr.
   (d) Since average cost per job = $W_s \times$ (Rs. 3.50) = Re. 0.70, cost per day = $8 \times 5 \times$ Re. 0.70 = Rs. 28 per day.]

4. Consider a single server queueing system with a Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour, and the maximum permissible number of calling units in the system is two. Derive the steady state probability distribution of the number of calling units in the system, and the, calculate the expected number in the system.

   Derive the formula $P_n = \dfrac{(1-\rho)\,\rho^n}{1-(\rho)^{N+1}}$ $(\rho \neq 1)$

   [**Hint.** Here $\lambda = 3$, $1/\mu = 25/100$, $N = 2$.

   $$P_n = \frac{(0.25)\,(0.75)^n}{1-(0.75)^3} = (0.43)\,(0.75)^n,\; P_0 = \frac{1-\rho}{1-\rho^{2+1}} = \frac{0.25}{1-(0.75)^3} = 0.43.$$

   $$L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n\,(\rho^n)\,P_0 = P_0\,(\rho + 2\rho^2) = 0.81.]$$

5. At a railway station, only one train is handled at a time. The railway yard is sufficient only for 3 trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

   [**Hint.** $\lambda = 6$, $\mu = 12$, $N = 3$. Find, $P_0 = 0.53$.

   Since $P_n = P_0\,(\lambda/\mu)^n$, find $P = 0.27$, $P_2 = 0.13$, $P_3 = 0.07$.

   Therefore $L_q = 1\,(0.27) + 2\,(0.13) + 3\,(0.07) = 0.74$.

   Thus the average number of trains in the queue is 0.74 and each train takes on an average 1/2 hour for getting service. Therefore, $W_q = (0.74)\,(1/12)$ hour = 3.8 minutes.]

6. A department store has a single cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. The average number of customers that can be processed by the cashier is 24 per hour. Assume that the conditions for use of the single-channel queueing model apply :
   (a) What is the probability that the cashier is idle ?
   (b) What is the average number of customers in the queueing system ?
   (c) What is the average time a customer spends in the system ?
   (d) What is the average number of customers in the queue ?
   (e) What is the average time a customer spends in the queue waiting for services ?

7. A single channel queueing system has Poisson arrivals and exponential service times. The mean arrival rate is 88 transactions per hour and the mean service rate is 23 per hour. Determine :
   (a) The average time a customer will wait in the system.
   (b) The average number of customers waiting in the queue.
   (c) The utilization factor of the system.

8. Workers come to a tool store room to enquire about the special tools (required by them) for a particular job. The average time between the arrivals is 60 seconds and the arrivals are assumed to be Poisson distribution. The average service time is 40 seconds. Determine :
   (a) average queue length,
   (b) average length of non-empty queue,
   (c) average number of workers in the system including the workers being attended,
   (d) mean waiting time of an arrival,
   (e) average waiting time of an arrival (workers) who waits. **[Virbhadra 2000]**

   [**Ans.** (a) 1.33 workers, (b) 3 workers, (c) 2 workers, (d) 1.33 minutes per worker, (e) 2 minutes per worker]

9. Problems arrive at a computing centre in Poisson fashion with a mean arrival rate of 25 per hour. The average computing job requires 2 minutes of terminal time. Calculate the following :

(a) Average number of problems waiting for the computer use.
(b) The per cent of times an arrival can walk right in without having to wait.

10. In a bank cheques are cashed at a single 'teller' counter. Customers arrive at the counter in a Poisson manner at an average rate of 30 customers per hour. The teller takes on an average a minute and a half to cash a cheque. The service time has been shown to be exponentially distributed.
(i) Calculate the percentage of time the teller is busy.
(ii) Calcualte the average time a customer is expected to wait.
[**Ans.** (i) 75%, (ii) 6 minutes.]

11. In a Tool Crib manned by a single Assistant the operators arrive at the tool crib at the rate of 10 per hour. Each operator needs 3 minutes on an average to be served. Find out the loss of production due to waiting of an operator in a shift of 8 hours if the rate of production is 100 units per shift.

12. In a bank with a single server, there are two chairs for waiting customers. On an average one customer arrives every 10 minutes and each customer takes 5 minutes for getting served. Making suitable assumptions, find :
(i) the probability that an arrival will get a chair to set down,
(ii) the probability that an arrival will have to stand, and
(iii) expected waiting time of a customer.
[**Ans.** (i) 7/8, (ii) 1/8, (iii) 5 minutes.]

13. In a service department manned by one server, on an average one customer arrives every 10 minutes. It has been found that each customer requires 6 minutes to be served. Find out :
(i) Average queue length, (ii) Average time spent in the system,
(iii) The probability that there would be two customers in the queue.
[**Ans.** (i) 0.9 customers, (ii) 15 minutes, (iii) 1.44%]

14. At Dr. Parachi's clinic, patients arrive at an average of 6 patients per hour. The clinic is attended to by Dr. Prachi herself. Some patients require only the repeat prescription, some come for minor check-up while some others require thorough inspection for the diagnosis. This takes the doctor six minutes per patient on the average. It can be assumed that arrivals follows a Poisson distribution and the doctor's inspecion time follows an exponential distribution. Determine :
(i) The per cent of times a patient can walk right inside the doctors cabin, without having to wait;
(ii) the average number of patient in Dr. Prachi's clinic;
(iii) the average numeber of patients waiting for their term, and
(iv) the average time a patient spends in the clinic.
[**Ans.** (i) 40%, (ii) 1½ patient, (iii) 0.90 patients, (iv) 15 minutes]

15. Customers arrive at a sales-counter maned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.
[**Ans.** (i) $\frac{20}{36\,(36-20)} \times 3600$ seconds, (ii) $\frac{1}{36-20} \times 3600$ seconds]

16. A fertilizer company distributes its products by trucks loaded at its only loading station. Both, company trucks and contractors' trucks are used for this purpose. It was found out that on an average every 5 minutes one truck arrived and the average loading time was 3 minutes. 40 per cent of the trucks belong to the contractors. Making suitable assumption determine :
(i)The probability that a truck has to wait. (ii) The waiting time of a truck that waits.
(iii)The expected waiting time of contractors' trucks per day.
[**Ans.** (i) 3/5, (ii) 1/8, (iii) $12 \times 12 \times \frac{40}{100} \times \frac{12}{20\,(20-22)}$]

17. What is queueing theory ? Describe the different types of costs involved in a queueing system. In what areas of management can queueing theory be applied successfully ? Give examples. **[JNTU (MCA III) 2004]**

18. Describe a single server waiting line model. Give an example of real life for each of the following queueing models :
(i) First come – First Served
(ii) Last come – First served
(iii) Random pick service
(iv) Customers stay only it served instanlly. **[JNTU (MCA III) 2004]**

MODEL OBJECTIVE QUESTIONS

1. Customer behaviour in which be moves from one queue to another in a multiple channel situation is
(a) balking. (b) reneging. (c) jockeying. (d) alternating.

2. Which of the following characteristics apply to queueing system
(a) Customer population. (b) Arrival process. (c) Both (a) and (b). (d) Neither (a) nor (b).

3. Which of the following is not a key operating characteristic for a queuing system?
(a) Utilization factor. (b) Per cent idle time.
(c) Average time spent for waiting in system and queue. (d) None of the above.

4. Priority queue discipline may be classified as
   (a) finite or infinite.
   (b) limited and unlimited.
   (c) pre-emptive or non-pre-emptive.
   (d) all of the above.
5. Which symbol describes the inter-arrival time distribution ?
   (a) *D*.
   (b) *M*.
   (c) *G*.
   (d) All of the above.
6. Which of the following relationships is not true
   (a) $W_s = W_q + 1/\mu$.
   (b) $L_s = \lambda W_s$.
   (c) $L_s = L_q + 1/\lambda$.
   (d) $L_q = \lambda W_q$.
7. The calling population is assumed to be infinite when
   (a) arrivals are independent of each other.
   (b) capacity of the system is infinite.
   (c) service rate is faster than the arrival rate.
   (d) all of the above.
8. Which of the cost estimates and performance measures are not used for economic analysis of a queuing system?
   (a) Cost per server per unit of time.
   (b) Cost per unit of time for a customer waiting in the system.
   (c) Average number of customers in the system.
   (d) Average waiting time of customers in the system.
9. A calling population is considered to be infinite when
   (a) all customers arrive atonce.
   (b) arrivals are independent of each other.
   (c) arrivals are dependent upon each other.
   (d) all of the above.
10. The cost of providing service in a queueing system decrease with
   (a) decreaed average waiting time in the queue.
   (b) decreased arrival rate.
   (c) increased arrival rate.
   (d) none of the above.

Answers

1. (c) **2.** (c) **3.** (d) **4.** (c) **5.** (d) **6.** (c) **7.** (a) **8.** (d) **9.** (b)
10. (d) .

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